

QUANTUM HAMILTONIAN REDUCTION OF W -ALGEBRAS AND CATEGORY \mathcal{O}

by

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Abstract

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W-algebras are a class of non-commutative algebras related to the classical universal enveloping algebras. They can be defined as a subquotient of $U(\mathfrak{g})$ related to a choice of nilpotent element e and compatible nilpotent subalgebra \mathfrak{m} . The definition is a quantum analogue of the classical construction of Hamiltonian reduction.

We define a quantum version of Hamiltonian reduction by stages and use it to construct intermediate reductions between different W-algebras $U(\mathfrak{g}, e)$ in type A. This allows us to express the W-algebra $U(\mathfrak{g}, e')$ as a subquotient of $U(\mathfrak{g}, e)$ for adjacent nilpotent elements $e' \geq e$. It also produces a collection of $(U(\mathfrak{g}, e), U(\mathfrak{g}, e'))$ -bimodules analogous to the generalised Gel'fand–Graev modules used in the classical definition of the W-algebra; these can be used to obtain adjoint functors between the corresponding module categories.

The category of modules over a W-algebra has a full subcategory defined in a parallel fashion to that of the Bernstein–Gel'fand–Gel'fand (BGG) category \mathcal{O} ; this version of category $\mathcal{O}(e)$ for W-algebras is equivalent to an infinitesimal block of \mathcal{O} by an argument of Milićić and Soergel. We therefore construct analogues of the translation functors between the different blocks of \mathcal{O} , in this case being functors between the categories $\mathcal{O}(e)$ for different W-algebras $U(\mathfrak{g}, e)$. This follows an argument of Losev, and realises the category $\mathcal{O}(e')$ as equivalent to a full subcategory of the category $\mathcal{O}(e)$ where $e' \geq e$ in the refinement ordering. Future work is to use this to provide an alternate categorification of $U(\mathfrak{sl}_2)$ along the lines of the work of Bernstein, Frenkel and Khovanov.

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Chapter 1

Introduction

In this chapter, we review the most important notions that will be used in the rest of the thesis: W-algebras and their relation to Slodowy slices, quantum Hamiltonian reduction by stages, category \mathcal{O} and categorification.

1.1 W-algebras

W-algebras, which we shall define in section 2.2.2, have been studied by a number of mathematicians and physicists over the past 25 years (cf. [dBT, Pre2]). The differing backgrounds and motivations of the researchers have produced several equivalent definitions – though not always obviously so – rooted in different fields and perspectives. Physicists find them interesting due to the fact that they arise in the study of conformal field theory, while representation theorists look to the insight they can give us into the classical representation theory of Lie algebras.

For a semisimple Lie algebra \mathfrak{g} , the universal enveloping algebra $U(\mathfrak{g})$ is an associative algebra which completely controls the representation theory of \mathfrak{g} . The category $U(\mathfrak{g})\text{-mod}$ is therefore of central concern to us. As in the study of the representations of any algebra, a lot of information can be determined by looking at how the centre $Z(\mathfrak{g}) := Z(U(\mathfrak{g}))$ acts. A central question to our work is what other algebras encapsulate important information about the representation theory of $U(\mathfrak{g})$? The W-algebras $U(\mathfrak{g}, e)$ form precisely one such class of algebras.

Given a semisimple Lie algebra \mathfrak{g} with a chosen nilpotent element e , the W-algebra $U(\mathfrak{g}, e)$ is an associative algebra which lies between the algebras $U(\mathfrak{g})$ and $Z(\mathfrak{g})$. More precisely, $U(\mathfrak{g}, e)$ is a subquotient of $U(\mathfrak{g})$ determined by the nilpotent element e , where $U(\mathfrak{g}, 0) = U(\mathfrak{g})$ and $U(\mathfrak{g}, e_{\text{reg}}) \simeq Z(\mathfrak{g})$ for the regular nilpotent $e_{\text{reg}} \in \mathfrak{g}$ (for example, the full Jordan block $J_n(0)$ in \mathfrak{sl}_n). This second statement was known to Kostant, and the modern definition of W-algebras in many ways seeks to generalise his earlier work on this special case [Kos]. Between these two extremes lies one isomorphism class of algebras for each nilpotent orbit in \mathfrak{g} . These intermediate W-algebras control the representation theory of the blocks of $U(\mathfrak{g})\text{-mod}$ corresponding to different central characters, where more singular nilpotent elements e will control blocks corresponding to less singular central characters.

Though this definition seems purely algebraic, it is actually a quantum version of the classical geometric idea of Hamiltonian reduction of Poisson varieties. Considering a nilpotent element e in a semisimple Lie algebra \mathfrak{g} associated to an algebraic group G , one can consider the nilpotent orbit $\mathcal{O}_e := G \cdot e$ (note that the notation for the nilpotent orbit \mathcal{O}_e is similar to the notation for category \mathcal{O} – which is meant in a given context will generally be clear). If e is

completed to an \mathfrak{sl}_2 -triple $\{e, h, f\}$, this gives rise to a natural transverse slice to \mathcal{O}_e known as the *Slodowy slice*, $\mathcal{S}_e = e + \ker \operatorname{ad} f$, for example as in fig. 1.1. We usually consider the Slodowy slice \mathcal{S}_e as a subset of \mathfrak{g}^* using the isomorphism $\mathfrak{g} \simeq \mathfrak{g}^*$ induced by the Killing form. In fact, the Slodowy slice can be expressed as the Hamiltonian reduction of \mathfrak{g}^* with respect to the action of a certain unipotent algebraic group depending on the \mathfrak{sl}_2 -triple. The subquotient definition for W-algebras can be understood in an analogous way.

$$e = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad h = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad f = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \mathcal{S}_e = \left\{ \begin{pmatrix} a & 0 & 0 \\ b & a & d \\ c & 0 & -2a \end{pmatrix} : a, b, c, d \in \mathbb{C} \right\}$$

Figure 1.1: The Slodowy slice of an \mathfrak{sl}_2 -triple in \mathfrak{sl}_3 .

The universal enveloping algebra $U(\mathfrak{g})$ is a filtered algebra whose associate graded algebra is the ring of functions $\mathbb{C}[\mathfrak{g}^*]$. Given that \mathfrak{g} is a Lie algebra, the ring of functions $\mathbb{C}[\mathfrak{g}^*]$ acquires a Poisson bracket coming from the Lie bracket on \mathfrak{g} , known as either the Lie–Poisson or the Kostant–Kirillov bracket. The multiplication in $U(\mathfrak{g})$ encodes both the multiplication and the Poisson bracket in $\mathbb{C}[\mathfrak{g}^*]$; this is known as a deformation quantisation of the Poisson algebra $\mathbb{C}[\mathfrak{g}^*]$. The W-algebra $U(\mathfrak{g}, e)$ has a similar geometric interpretation: its associate graded algebra is the ring of functions on the Slodowy slice $\mathcal{S}_e \subseteq \mathfrak{g}^*$. Further, $U(\mathfrak{g}, e)$ is the unique deformation quantisation (up to isomorphism) of the ring of functions on the Slodowy slice \mathcal{S}_e . The structure of the W-algebra $U(\mathfrak{g}, e)$ is therefore geometrically encoded in the Poisson variety \mathcal{S}_e [GG].

This geometric interpretation allows us to better understand the definition of the W-algebra: upon passing to the associate graded algebra, expressing $U(\mathfrak{g}, e)$ as a subquotient of $U(\mathfrak{g})$ corresponds to choosing expressing \mathcal{S}_e as a Hamiltonian reduction of \mathfrak{g}^* . Correspondingly, the subquotient definition of the W-algebra can be understood as expressing $U(\mathfrak{g}, e)$ as a *quantum Hamiltonian reduction* (QHR) of $U(\mathfrak{g})$. In this interpretation, the nilpotent element e plays the role of a regular value of the moment map, and a certain nilpotent Lie algebra \mathfrak{m} (the Premet subalgebra) associated to e plays the role of the unipotent subgroup acting on \mathfrak{g}^* .

1.2 Quantum Hamiltonian reduction by stages

It can be taken as the definition of W-algebras that they can be expressed as certain quantum Hamiltonian reductions of $U(\mathfrak{g}) = U(\mathfrak{g}, 0)$, but one might ask whether a W-algebra $U(\mathfrak{g}, e')$ can be expressed as a QHR of another W-algebra $U(\mathfrak{g}, e)$ in a way compatible with the original reduction – that is, do there exist pairs of nilpotent elements e and e' for which we can express $U(\mathfrak{g}, e')$ as a QHR of $U(\mathfrak{g}, e)$ in such a way that fig. 1.2 commutes up to isomorphism? Upon dequantising the diagram, this would correspond to expressing $\mathcal{S}_{e'}$ as a Hamiltonian reduction of \mathfrak{g}^* by stages, passing through the Slodowy slice \mathcal{S}_e in the middle step. The theory of Hamiltonian reduction by stages is a well-developed branch of symplectic geometry, and has been outlined in – for example – [MMO⁺]. To construct our intermediate reductions, we therefore need to develop a quantum version of Hamiltonian reduction by stages.

The first main result of this thesis is conjecture 1.2.1, which states that we can construct such an intermediate set of reductions in Lie algebras of type A for all pairs of nilpotent elements such that e' covers e in the dominance ordering, that is whenever $\mathcal{O}_e \subseteq \overline{\mathcal{O}_{e'}}$ and for which there

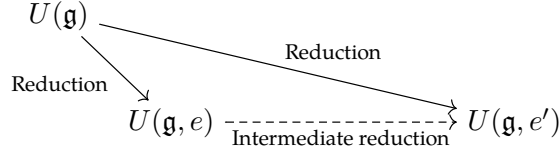


Figure 1.2: Reduction of W-algebras by stages.

are no intermediate orbits. The result is to construct a sequence of reductions for all W-algebras corresponding to nilpotent elements of the Lie algebra \mathfrak{g} , such that $U(\mathfrak{g}, e)$ can be reduced to $U(\mathfrak{g}, e')$ precisely if e' covers e . In such a way, a sequence of commuting reductions is obtained which is precisely the reverse of the Hasse diagram for nilpotents orbits under the dominance ordering (see fig. 1.3, for example).

Conjecture 1.2.1. *Let \mathfrak{g} be a semisimple Lie algebra of type A, and e and e' be two nilpotent elements such that e' covers e in the dominance ordering. Then there exists a Lie subalgebra $\mathfrak{k} \subseteq \mathfrak{g}$ and a left ideal $U(\mathfrak{g}, e)\mathfrak{k}_\kappa \subseteq U(\mathfrak{g}, e)$ such that*

$$U(\mathfrak{g}, e') \simeq (U(\mathfrak{g}, e)/U(\mathfrak{g}, e)\mathfrak{k}_\kappa)^\mathfrak{k},$$

where invariants are taken with respect to the adjoint action of \mathfrak{k} .

To construct the reductions, we use the theory of *pyramids* in semisimple Lie algebras developed by Elashvili and Kac [EK]. Pyramids are combinatorial objects closely related to Young tableaux, however they encode not only the nilpotent element e , but also a *good grading* of \mathfrak{g} for e . This allows much of the same information provided by an \mathfrak{sl}_2 -triple to be specified by a weaker piece of data, giving us extra flexibility needed for this construction. This provides a construction which produces a quantum Hamiltonian reduction of the W-algebra $U(\mathfrak{g}, e)$, which conjecturally is isomorphic to the W-algebra $U(\mathfrak{g}, e')$.

1.3 Category \mathcal{O}

For a semisimple Lie algebra \mathfrak{g} , there is a well-studied category of its representations known as category \mathcal{O} . It consists of all representations satisfying a certain finiteness condition, and in particular contains all finite-dimensional modules. Category \mathcal{O} has a number of remarkable properties, and decomposes naturally into blocks in a way that allows it to be equipped with a number of module structures that parallel important classical constructions (cf. [BFK, KMS1, KMS2]): this kind of enrichment is known as a *categorification*. The representations of W-algebras are naturally related to the blocks of category \mathcal{O} , and in my work I attempt to use this relation to construct categorifications of classical objects.

The representation theory of $U(\mathfrak{g}, e)$ has been studied by a number of authors, and in particular its relationship to the representation theory of $U(\mathfrak{g})$ has been examined in [BGK, Web, Los3]. For example, Skryabin has shown that the category $U(\mathfrak{g}, e)\text{-mod}$ is equivalent to a full subcategory of $U(\mathfrak{g})\text{-mod}$, characterised by a certain finiteness property with respect to the action of a subalgebra $U(\mathfrak{m})$ determined by e [Pre1].

Given a semisimple Lie algebra \mathfrak{g} , a choice of triangular decomposition $\mathfrak{g} \simeq \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$ allows us to define the highest weight modules as those which are a finite union of sets of the form $U(\mathfrak{n}_-) \cdot v$ for v a *highest weight vector* (i.e. $\mathfrak{n}_+ \cdot v = 0$). Many important $U(\mathfrak{g})$ -modules

are highest weight modules, including all finite-dimensional representations and Verma modules. The *BGG category* \mathcal{O} [BGG] is the minimal full subcategory of $U(\mathfrak{g})\text{-mod}$ containing all highest-weight modules that is closed under direct sums, sub- and quotient modules, kernels and cokernels, and tensoring with finite-dimensional modules (the so-called *translation functors*). This remains rich enough to contain a lot of information about the representation theory of $U(\mathfrak{g})$, while having additional properties making it more amenable to study. In particular it is a *highest weight category*, which allows us to choose a number of convenient bases for its Grothendieck group (that is, the group whose elements are formal differences of isomorphism classes of representations and whose group operation is the direct sum of representations).

Category \mathcal{O} has a natural *block decomposition* $\mathcal{O} \simeq \bigoplus \mathcal{O}_\chi$ indexed by the generalised central characters of $U(\mathfrak{g})$. There are no homomorphisms or non-trivial extensions between modules belonging to different blocks, so to understand the structure of category \mathcal{O} it suffices to understand the blocks \mathcal{O}_χ . In [MS], Milićić and Soergel apply the theory of Harish-Chandra bimodules to construct an equivalence of categories exchanging the condition on the (generalised) central character with a related condition on the (generalised) nilpotent character. We choose a nilpotent element e compatible with χ in the sense that the stabiliser subgroup of χ under the ‘dotted Weyl action’ is generated by the simple reflections corresponding to e . When combined with the Skryabin equivalence, this can be used to construct an equivalence $\mathcal{O}_\chi \simeq \mathcal{O}_0(e)$, where $\mathcal{O}_0(e)$ is a full subcategory of $U(\mathfrak{g}, e)\text{-mod}$ analogous to a regular block of category \mathcal{O} . [Los1, Web]

The construction of quantum Hamiltonian reduction in conjecture 1.2.1 produces a pair of adjoint functors $U(\mathfrak{g}, e)\text{-mod} \rightleftarrows U(\mathfrak{g}, e')\text{-mod}$, however these functors do not preserve the finiteness conditions of the categories $\mathcal{O}(e)$ and $\mathcal{O}(e')$. To solve this problem we adapt one of the techniques of Losev in [Los1], in which he proves that $\mathcal{O}(e)$ is equivalent to a full subcategory of $U(\mathfrak{g})\text{-mod}$ called the Whittaker category. This allows us to realise the category $\mathcal{O}(e')$ as a full subcategory of $U(\mathfrak{g}, e)\text{-mod}$. Future work is to use this embedding along with *averaging functors* to construct pairs of functors $\mathcal{O}(e) \rightleftarrows \mathcal{O}(e')$, analogous to the classical translation functors between different infinitesimal blocks of category \mathcal{O} . We hope in the future to show that these functors intertwine with the translation functors through the Milićić–Soergel equivalence.

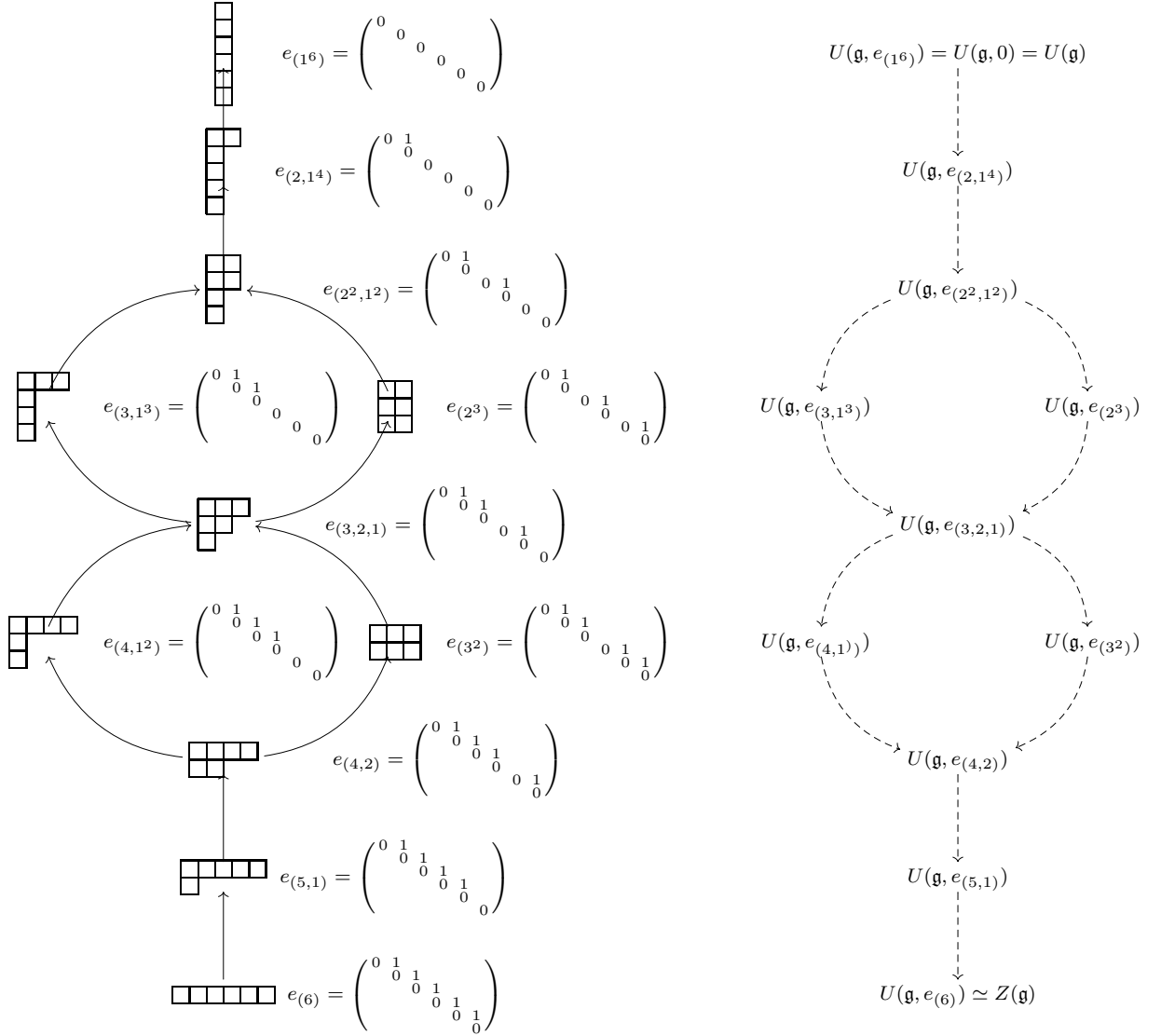


Figure 1.3: The Hasse diagram and intermediate quantum Hamiltonian reductions for $\mathfrak{g} = \mathfrak{sl}_6$. Each partition P of 6 has a corresponding conjugacy class of nilpotent matrices – here their representative e_P in Jordan canonical form is shown to the right of the corresponding partition – and each has an associated W-algebra $U(\mathfrak{g}, e_P)$. The diagram showing the intermediate quantum Hamiltonian reductions between W-algebras is the reverse of the Hasse diagram.

Chapter 2

W-algebras

In this chapter, we will work towards defining the basic objects of our study: the W-algebras $U(\mathfrak{g}, e)$. There are a number of equivalent definitions of W-algebras given in a number of different sources (cf. [Wan, §3]). We shall generally use a definition of W-algebras expressed as a certain subquotient of the universal enveloping algebra $U(\mathfrak{g})$ known as the Whittaker module definition, though we'll occasionally remark upon and use the equivalence with other formulations.

2.1 Nilpotent orbits and Slodowy slices

We begin by recalling some basic facts about the nilpotent cone $\mathcal{N} \subseteq \mathfrak{g}$. The algebraic group G acts on \mathfrak{g} by the adjoint action, and this action preserves the nilpotent cone. As a result, we have a stratification of \mathcal{N} into nilpotent orbits \mathcal{O}_e , where $\mathcal{O}_e := G \cdot e$ is the orbit of the nilpotent element e under the adjoint action of G . This holds in an arbitrary semisimple Lie algebra, but it has a particularly simple form in type A, where $\mathfrak{g} = \mathfrak{sl}_n$: $G = SL_n$ acts on \mathfrak{g} by conjugation and the nilpotent cone \mathcal{N} consists of all nilpotent matrices in \mathfrak{g} , which are classified up to conjugacy by their Jordan canonical form.

The nilpotent cone \mathcal{N} has a unique dense open orbit \mathcal{O}_{reg} called the *regular orbit*, which consists of all *regular* nilpotent elements: that is elements $e \in \mathfrak{g}$ for which $\dim Z_G(e) = \text{rank } \mathfrak{g}$, where $Z_G(e)$ is the stabiliser of e in G . The complement of the regular orbit $\mathcal{N} \setminus \mathcal{O}_{\text{reg}}$ itself has a unique open dense orbit \mathcal{O}_{sub} called the *subregular orbit*, and a unique *minimal orbit* \mathcal{O}_{min} of smallest strictly-positive dimension. In type A_{n-1} these special orbits have explicit descriptions in terms of the Jordan canonical form of the nilpotent elements comprising them:

- \mathcal{O}_{reg} has elements with a single Jordan block of size n .
- \mathcal{O}_{sub} has elements with one block of size $n - 1$ and another of size 1.
- \mathcal{O}_{min} has elements with one block of size 2 and $n - 2$ blocks of size 1.

The set of nilpotent orbits in \mathcal{N} is naturally a partially ordered set, where $\mathcal{O}' \leq \mathcal{O}$ if and only if $\mathcal{O}' \subseteq \overline{\mathcal{O}}$. Under this ordering, we can make some statements about the three nilpotent orbits described above. The orbit \mathcal{O}_{reg} is the maximal element of the poset, \mathcal{O}_{sub} is the maximal element of the poset lying under \mathcal{O}_{reg} , and \mathcal{O}_{min} is the minimal element of the poset lying above the zero orbit $\{0\}$.

2.1.1 The Jacobson–Morozov theorem

In order to discuss nilpotent orbits, it will frequently be convenient to complete a given nilpotent element $e \in \mathfrak{g}$ to an \mathfrak{sl}_2 -triple $\{e, h, f\} \subseteq \mathfrak{g}$, i.e. e, h and f satisfy the \mathfrak{sl}_2 commutation relations $[h, e] = 2e$, $[h, f] = -2f$, and $[e, f] = h$. Another way of phrasing this condition is that there exists a Lie algebra homomorphism $\mathfrak{sl}_2 \rightarrow \mathfrak{g}$ such that the image of e in the standard basis of \mathfrak{sl}_2 is the element $e \in \mathfrak{g}$. The *Jacobson–Morozov theorem* states that it is always possible to extend a nilpotent element e to an \mathfrak{sl}_2 -triple in a semisimple Lie algebra. There are a number of different proofs of this theorem in the literature (cf. [CG, §3.7.25]); the version we present here comes from [CM, §3.3].

The Jacobson–Morozov theorem. *For any non-zero nilpotent element e in a semisimple Lie algebra \mathfrak{g} , there exists an \mathfrak{sl}_2 -triple $\{e, h, f\} \subseteq \mathfrak{g}$ such that $[h, e] = 2e$, $[h, f] = -2f$, and $[e, f] = h$.*

Proof. We prove this by induction on $\dim \mathfrak{g}$. The non-zero semisimple Lie algebra with smallest possible dimension is \mathfrak{sl}_2 itself, and since any non-zero nilpotent in \mathfrak{sl}_2 is conjugate to any other, we can simply conjugate the standard \mathfrak{sl}_2 -triple to coincide with e . We now proceed with the inductive step. If e lies in a proper semisimple subalgebra of \mathfrak{g} , then we can apply the inductive hypothesis, so we now assume that e lies in no proper semisimple subalgebra of \mathfrak{g} .

We first prove that $\langle e, \mathfrak{z}(e) \rangle = 0$. Consider $x \in \mathfrak{z}(e)$, and recall that $\langle e, x \rangle = \text{tr}(\text{ad } e \text{ ad } x)$. The Jacobi identity shows that $\text{ad } e$ commutes with $\text{ad } x$ for any $x \in \mathfrak{z}(e)$, so we can state that $(\text{ad } e \text{ ad } x)^n = (\text{ad } e)^n (\text{ad } x)^n$ for any $n \in \mathbb{N}$. The element e is nilpotent, and so $(\text{ad } e)^n = 0$ for large enough n ; as a result, the operator $\text{ad } e \text{ ad } x$ is also nilpotent and hence traceless, proving our claim.

We thus know that $e \in (\mathfrak{z}(e))^\perp$, where the orthogonal complement is taken with respect to the Killing form; we now show that $(\mathfrak{z}(e))^\perp = [\mathfrak{g}, e]$. First note that the associativity of the Killing form shows that $[\mathfrak{g}, e] \subseteq (\mathfrak{z}(e))^\perp$; to show that this is everything we count dimensions. Note that the map $\text{ad } e: \mathfrak{g} \rightarrow \mathfrak{g}$ has kernel $\mathfrak{z}(e)$ and image $[\mathfrak{g}, e]$, and so the rank–nullity theorem tells us that $\dim[\mathfrak{g}, e] = \dim \mathfrak{g} - \dim \mathfrak{z}(e)$. This is precisely the dimension of $(\mathfrak{z}(e))^\perp$. As a result, we know that $[h, e] = 2e$ for some element $h \in \mathfrak{g}$. We can further take h to be semisimple, as if it were not its semisimple part would also satisfy this property.

Lemma 2.1.1. *The element h constructed above lies in $[\mathfrak{g}, e]$.*

Assuming this lemma for the moment, we give a proof of the Jacobson–Morozov theorem. Let $f' \in \mathfrak{g}$ be an element such that $[e, f'] = h$; since h is semisimple we have that $\mathfrak{g} = \bigoplus_j \mathfrak{g}_{\lambda_j}$ where $\mathfrak{g}_{\lambda_j} = \{x \in \mathfrak{g} : [h, x] = \lambda_j x\}$ is the eigenspace corresponding to eigenvalue λ_j . We therefore have a decomposition $f' = \sum_j f_j$, where f_j is the projection of f' onto \mathfrak{g}_{λ_j} : this gives us that $h = \sum_j [e, f_j]$. However, we can note that $h \in \mathfrak{g}_0$, and that $[e, \mathfrak{g}_{\lambda_j}] \subseteq \mathfrak{g}_{\lambda_j+2}$; this allows us to see that there exists a k with $\lambda_k = -2$, and that $h = [e, f_k]$. Taking $f = f_k$ gives us our \mathfrak{sl}_2 -triple $\{e, h, f\}$, and completes the proof of the Jacobson–Morozov theorem.

It remains to prove lemma 2.1.1. We shall prove this by contradiction: we shall assume that $h \notin [\mathfrak{g}, e]$, and construct a proper semisimple subalgebra of \mathfrak{g} containing e , violating the assumption we made at the beginning of the proof preventing us from using the inductive hypothesis. Since $[\mathfrak{g}, e] = (\mathfrak{z}(e))^\perp$, we know that $\langle h, \mathfrak{z}(e) \rangle \neq 0$.

Since $\text{ad } h$ preserves $\mathfrak{z}(e)$, we can decompose it into $\text{ad } h$ eigenspaces $\bigoplus_j \mathfrak{z}(e)_{\mu_j}$, where $\mathfrak{z}(e)_0$ is the centraliser of h in $\mathfrak{z}(e)$: this gives us the decomposition $\mathfrak{z}(e) = \mathfrak{z}(e)_0 \oplus \bigoplus_{\mu_j \neq 0} \mathfrak{z}(e)_{\mu_j}$. Associativity of the Killing form tells us that $\langle h, [h, \mathfrak{z}(e)] \rangle = 0$, while the eigenvalue decomposition tells us that for $x \in \mathfrak{z}(e)_{\mu_j}$ we have that $\langle h, [h, x] \rangle = \mu_j \langle h, x \rangle$. We therefore know that $h \in (\mathfrak{z}(e)_{\mu_j})^\perp$ for any $\mu_j \neq 0$. So to satisfy the condition that $\langle h, \mathfrak{z}(e) \rangle \neq 0$, there must exist an

element $z \in \mathfrak{z}(e)_0 = \mathfrak{z}_{\mathfrak{z}(e)}(h)$ with $\langle h, z \rangle \neq 0$. We can assume that z is semisimple by the same argument used above to prove that there exists a semisimple element h with $[h, e] = 2e$.

Since the centraliser of any semisimple element is reductive (cf. [CM, Lemma 2.1.2]), we have that $[\mathfrak{z}(z), \mathfrak{z}(z)]$ is a semisimple subalgebra of \mathfrak{g} . It is proper, since no non-zero semisimple element commutes with all of \mathfrak{g} (by the definition of semisimplicity). By construction, z commutes with both h and e , and hence $2e = [h, e] \in [\mathfrak{z}(z), \mathfrak{z}(z)]$. Thus $[\mathfrak{z}(z), \mathfrak{z}(z)]$ is a proper semisimple subalgebra of \mathfrak{g} containing e , contradicting our hypothesis. This completes the proof of lemma 2.1.1, and of the Jacobson–Morozov theorem. \square

Example 2.1.2. The proof of the Jacobson–Morozov theorem is unfortunately non-constructive, however in type A we can provide an explicit construction of an \mathfrak{sl}_2 -triple for a given nilpotent e . We can always conjugate e into Jordan canonical form, and it suffices to give an \mathfrak{sl}_2 -triple for a Jordan block: the \mathfrak{sl}_2 -triple for the full Jordan canonical form can be constructed from the given blocks.

For a nilpotent consisting of a single Jordan block of size n , an \mathfrak{sl}_2 -triple consists of the elements $h = \text{diag}(n-1, n-3, \dots, 1-n)$ and

$$f = \begin{pmatrix} 0 & & & 0 \\ a_1 & 0 & & \\ & \ddots & \ddots & \\ 0 & & a_{n-1} & 0 \end{pmatrix},$$

where $a_i = i(n-i)$.

2.1.2 Good gradings

An \mathfrak{sl}_2 -triple $\{e, h, f\}$ contains a semisimple element h , and as a result \mathfrak{g} will decompose into a direct sum of eigenspaces of the operator $\text{ad } h$. Specifically we can write $\mathfrak{g} = \bigoplus_{j \in \mathbb{Z}} \mathfrak{g}_j$, where $\mathfrak{g}_j = \{x \in \mathfrak{g} : [h, x] = jx\}$. The Jacobi identity implies that $[\mathfrak{g}_i, \mathfrak{g}_j] \subseteq \mathfrak{g}_{i+j}$, and so any \mathfrak{sl}_2 -triple endows \mathfrak{g} with a natural \mathbb{Z} -grading. In fact, every \mathbb{Z} -grading comes from the action of a semisimple element in such a way.

Lemma 2.1.3. *Given a \mathbb{Z} -grading $\Gamma: \mathfrak{g} = \bigoplus_{j \in \mathbb{Z}} \mathfrak{g}_j$, there exists a semisimple element $h_\Gamma \in \mathfrak{g}$ such that $\mathfrak{g}_j = \{x \in \mathfrak{g} : [h_\Gamma, x] = jx\}$.*

Proof. Note that the degree map $\partial: \mathfrak{g} \rightarrow \mathbb{Z}$ given by $\partial(x) = jx$ for $x \in \mathfrak{g}_j$ is a derivation of the semisimple Lie algebra \mathfrak{g} . Since all derivations of a semisimple Lie algebra are inner derivations, there exists a semisimple element $h_\Gamma \in \mathfrak{g}$ such that $\partial = \text{ad } h_\Gamma$. \square

Remark 2.1.4. This proof can be extended to reductive Lie algebras such as \mathfrak{gl}_n .

We will be interested in the properties of \mathbb{Z} -gradings coming from \mathfrak{sl}_2 -triples, and so we will give them a special name: \mathbb{Z} -gradings coming from \mathfrak{sl}_2 -triples by the above procedure are called *Dynkin gradings*. Dynkin gradings have a number of useful properties:

GG1. $e \in \mathfrak{g}_2$,

GG2. $\text{ad } e: \mathfrak{g}_j \rightarrow \mathfrak{g}_{j+2}$ is injective for $j \leq -1$,

GG3. $\text{ad } e: \mathfrak{g}_j \rightarrow \mathfrak{g}_{j+2}$ is surjective for $j \geq -1$,

$$\text{GG4. } \mathfrak{z}(e) \subseteq \bigoplus_{j \geq 0} \mathfrak{g}_j,$$

$$\text{GG5. } \langle \mathfrak{g}_i, \mathfrak{g}_j \rangle = 0 \text{ unless } i + j = 0,$$

$$\text{GG6. } \dim \mathfrak{z}(e) = \dim \mathfrak{g}_0 + \dim \mathfrak{g}_1.$$

Property GG1 follows directly from the definition of an \mathfrak{sl}_2 -triple, while properties GG2 and GG3 follow from the fact that \mathfrak{g} has the structure of a finite-dimensional \mathfrak{sl}_2 -representation, and hence decomposes as a direct sum of irreducible \mathfrak{sl}_2 -modules (see fig. 2.1). Properties GG4, GG5 and GG6, on the other hand, can be proven from properties GG1, GG2 and GG3 directly.

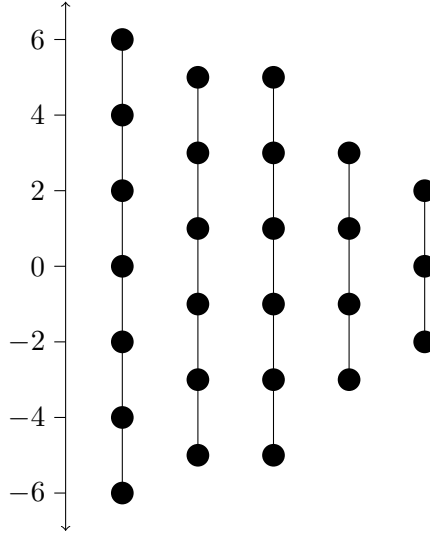


Figure 2.1: The decomposition of a finite-dimensional \mathfrak{sl}_2 -module as a sum of simple modules. Each column is an irreducible component, and each dot represents the 1-dimensional weight space of weight given at the left. The action of h corresponds to scaling by the weight, the action of e moves up the string to the next weight space and the action of f moves down.

Proposition 2.1.5. *Any \mathbb{Z} -grading satisfying properties GG1, GG2 and GG3 will also satisfy properties GG4, GG5 and GG6.*

Proof. Note that property GG2 implies that $\mathfrak{z}(e) = \ker \text{ad } e$ must not have any component in \mathfrak{g}_j for $j \leq -1$, whence property GG4. To prove property GG5, we consider the semisimple element h_Γ coming from lemma 2.1.3: taking $x \in \mathfrak{g}_i$ and $y \in \mathfrak{g}_j$ we can note that $\langle [x, h], y \rangle = \langle x, [h, y] \rangle$, and hence $-i\langle x, y \rangle = j\langle x, y \rangle$. As a result, either $i + j = 0$ or $\langle x, y \rangle = 0$, proving property GG5. Property GG6 follows from the following sequence, which is short exact by properties GG3 and GG4:

$$0 \longrightarrow \mathfrak{z}(e) \longrightarrow \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_{>0} \xrightarrow{\text{ad } e} \mathfrak{g}_{>0} \longrightarrow 0,$$

where $\mathfrak{g}_{>0} = \bigoplus_{j \geq 0} \mathfrak{g}_j$. Dimension counting then gives that $\dim \mathfrak{z}(e) = \dim \mathfrak{g}_{-1} + \dim \mathfrak{g}_0$, which gives property GG6 when combined with the fact that $\text{ad } e: \mathfrak{g}_{-1} \rightarrow \mathfrak{g}_1$ is a bijection from properties GG2 and GG3. \square

Remark 2.1.6. We can actually make a stronger statement. Note that property GG5 holds for any \mathbb{Z} -grading Γ ; we can therefore use property GG5 and the non-degeneracy of the Killing form to prove that properties GG2 and GG3 are equivalent for any \mathbb{Z} -grading Γ .

It turns out that we need slightly more flexibility than the Dynkin gradings are able to provide us, but the above proposition tells us that many of the desirable properties of Dynkin gradings can be obtained from arbitrary \mathbb{Z} -gradings which satisfy properties GG1 to GG3. This motivates the following definition.

Definition 2.1.7. Given a semisimple Lie algebra \mathfrak{g} with chosen nilpotent element e , a \mathbb{Z} -grading $\Gamma: \mathfrak{g} = \bigoplus_{j \in \mathbb{Z}} \mathfrak{g}_j$ is called a *good grading* for e if it satisfies properties GG1 to GG3.

Remark 2.1.8. Lemma 2.1.3 implies that any good grading Γ for a nilpotent $e \in \mathfrak{g}$ can be expressed as the eigenspaces of a semisimple element h_Γ for which $[h_\Gamma, e] = 2e$; however, this does not mean that any good grading comes from an \mathfrak{sl}_2 -triple $\{e, h_\Gamma, f\}$. In particular, there may not exist an element f which completes $\{e, h_\Gamma\}$ to an \mathfrak{sl}_2 -triple. Hence, though every Dynkin grading is a good grading, there exist good gradings which are not Dynkin, as we shall see later in section 3.2.

Note. A \mathbb{Z} -grading Γ of \mathfrak{g} is said to be *good* if there exists a nilpotent $e \in \mathfrak{g}$ for which it is a good grading; further, such a nilpotent is *good* for Γ . The \mathbb{Z} -grading Γ is said to be an *even grading* if $\mathfrak{g}_{2j+1} = \{0\}$ for any integer j .

Definition 2.1.9. Let Γ be a good grading for the nilpotent element $e \in \mathfrak{g}$. A Γ -graded \mathfrak{sl}_2 -triple is an \mathfrak{sl}_2 -triple $\{e, h, f\}$ such that $e \in \mathfrak{g}_2$, $h \in \mathfrak{g}_0$ and $f \in \mathfrak{g}_{-2}$.

Note. It should again be emphasised that a Γ -graded \mathfrak{sl}_2 -triple is only compatible with the good grading Γ in the sense outlined in the definition. In particular, the Dynkin grading coming from $\text{ad } h$ is not generally the same as Γ .

Lemma 2.1.10. For any non-zero nilpotent $e \in \mathfrak{g}$ and good \mathbb{Z} -grading Γ , there exists a Γ -graded \mathfrak{sl}_2 -triple $\{e, h, f\}$.

Proof. By the Jacobson–Morozov theorem, the nilpotent e can be completed to an \mathfrak{sl}_2 -triple $\{e, h', f'\}$. Let $h' = \sum_{j \in \mathbb{Z}} h_j$ and $f' = \sum_{j \in \mathbb{Z}} f'_j$ be the decompositions with respect to Γ , and define $h = h_0$: it follows that $[h, e] = 2e$ and $h = [e, f'_{-2}]$. Next, construct \tilde{f} as the component of f'_{-2} which lies in the -2 eigenspace of $\text{ad } h$. We now have an \mathfrak{sl}_2 -triple $\{e, h, \tilde{f}\}$, though it may no longer be the case that $\tilde{f} \in \mathfrak{g}_{-2}$; however, taking f to be the -2 component of \tilde{f} with respect to Γ provides the required \mathfrak{sl}_2 -triple. In fact we note that $[e, f - \tilde{f}] = 0$, which implies that $f = \tilde{f}$ by property GG2. \square

The characteristic of a grading

We will be interested in determining what the possible good gradings are in a given Lie algebra. The answer is known, though quite complicated in general; however, we can make some preliminary remarks greatly narrowing down the possibilities. We begin by defining the *characteristic* of a \mathbb{Z} -grading. We note that \mathfrak{g}_0 is a reductive subalgebra of \mathfrak{g} , and a Cartan subalgebra \mathfrak{h} of \mathfrak{g}_0 is also a Cartan subalgebra of \mathfrak{g} ; we consider the root space decomposition $\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha} \mathfrak{g}_{\alpha}$. Let Δ_0^+ be a system of positive roots of pure degree in \mathfrak{g}_0 ; the set $\Delta^+ := \Delta_0^+ \cup \{\alpha : \mathfrak{g}_{\alpha} \subseteq \mathfrak{g}_{>0}\}$ forms a system of positive roots in \mathfrak{g} . Choose a set Φ of simple roots in Δ^+ and let $\Phi_j := \Phi \cap \mathfrak{g}_j$ for each $j \geq 0$.

Definition 2.1.11. The *characteristic* of a \mathbb{Z} -grading is the decomposition $\Phi = \bigcup_{j \geq 0} \Phi_j$.

We note that there is a bijection between the set of \mathbb{Z} -gradings on \mathfrak{g} up to conjugation and the set of all possible characteristics.

Proposition 2.1.12. *If Γ is a good grading for a nilpotent e , then $\Phi = \Phi_0 \cup \Phi_1 \cup \Phi_2$.*

Proof. Let $\Phi = \{\alpha_1, \dots, \alpha_r\}$, and assume there exists some simple root $\alpha_j \notin \Phi_0 \cup \Phi_1 \cup \Phi_2$; α_j must therefore lie in Φ_k for some $k > 2$. Let e_α be a generator of the weight space \mathfrak{g}_α . Since $e \in \mathfrak{g}_2$, it must lie in the subalgebra generated by $\{\alpha_i : i \neq j\}$. Hence $[e, e_{-\alpha_j}] = 0$, which violates property **GG2**. \square

Corollary 2.1.13. *If Γ is an even good grading for a nilpotent e , then $\Phi = \Phi_0 \cup \Phi_2$.*

Thus we can conclude that there are only a finite number of good gradings possible up to conjugacy: the total number is bounded by $3^{\text{rank } \mathfrak{g}}$. Unfortunately this bound is not sharp, and there are in general many fewer good gradings than would be suggested here.

2.1.3 A bijection between nilpotent orbits and \mathfrak{sl}_2 -triples

Our objective in this section has been to introduce the tools necessary for the study of nilpotent orbits in \mathcal{N} . The Jacobson–Morozov theorem states that any nilpotent element e can be completed to an \mathfrak{sl}_2 -triple, however it doesn't tell us how many 'different' such triples there are: we don't know whether the triple is 'essentially unique', or if there a number of different 'inequivalent' \mathfrak{sl}_2 -triples possible. More concretely, we consider two \mathfrak{sl}_2 -triples equivalent if they are conjugate (i.e. in the same orbit) under the adjoint action of G . We can then construct a map from the set of equivalence classes of \mathfrak{sl}_2 -triples to the set of non-zero nilpotent orbits in \mathfrak{g} .

$$\begin{aligned} \Omega: \{\mathfrak{sl}_2\text{-triples in } \mathfrak{g}\}/G &\rightarrow \{\text{non-zero nilpotent orbits in } \mathfrak{g}\} \\ [\{e, h, f\}] &\mapsto \mathcal{O}_e \end{aligned}$$

Theorem 2.1.14 (Kostant). *The map Ω is a bijection.*

This theorem tells us that when considering nilpotent orbits in \mathfrak{g} , we are completely justified in constructing \mathfrak{sl}_2 -triples, as there is a unique conjugacy class of \mathfrak{sl}_2 -triples for each orbit. Any constructions we make for a nilpotent orbit can be made using a choice of \mathfrak{sl}_2 -triple without worrying about different choices yielding different results.

Proof. [CM, §3.4] The map Ω is well-defined, as two conjugate \mathfrak{sl}_2 -triples will have elements e lying in the same nilpotent orbit. The fact that it is surjective follows directly from the Jacobson–Morozov theorem. It remains only to show that Ω is injective.

Assume we have two \mathfrak{sl}_2 -triples with the same image under Ω ; without loss of generality, we can conjugate so the triples have the form $\{e, h, f\}$ and $\{e, h', f'\}$. Consider the Lie algebra $\mathfrak{u}_e := \mathfrak{z}(e) \cap [\mathfrak{g}, e]$: we note that \mathfrak{u}_e is an $\text{ad } h$ -invariant ideal of $\mathfrak{z}(e)$, and that $\mathfrak{u}_e = \mathfrak{z}(e)_{>0}$. This second identity follows from property **GG4**, and the fact that $[\mathfrak{g}, e] \cap \mathfrak{g}_0 = [e, \mathfrak{g}_{-2}]$, and hence does not commute with e by \mathfrak{sl}_2 representation theory (see e.g. fig. 2.1). That \mathfrak{u}_e lies in strictly positive degree further implies that it is a nilpotent ideal of $\mathfrak{z}(e)$.

Let U_e be the connected subgroup of G with Lie algebra \mathfrak{u}_e . Since \mathfrak{u}_e is nilpotent the exponential map is a diffeomorphism, and the adjoint action has a particularly simple expression: for $x \in \mathfrak{u}_e$ and sufficiently large n ,

$$\exp(x) \cdot h = h + [x, h] + \frac{1}{2}[x, [x, h]] + \dots + \frac{1}{n!}[x, [x, \dots, [x, h]]].$$

The fact that \mathfrak{u}_e is $\text{ad } h$ -invariant implies that $U_e \cdot h \subseteq h + \mathfrak{u}_e$. We can show that this inclusion is in fact an equality either by constructing an appropriate element of U_e directly (cf. [CM,

Lemma 3.4.7]), or by observing that the orbit of the unipotent group is a Zariski-closed dense subset of $h + \mathfrak{u}_e$, and hence $h + \mathfrak{u}_e$ itself (cf. [CG, Lemma 3.7.21]).

We can now use the fact that $h - h' \in \mathfrak{u}_e$ to see that $h' \in U_e \cdot h$, and hence have proven that there exists an element of $U_e \subseteq G$ which fixes e under the adjoint action and sends h to h' . This allows us to conjugate $\{e, h', f'\}$ to $\{e, h, f''\}$, and the fact that $[e, f - f''] = 0$ implies that $f = f''$ by property GG2, thus completing the proof. \square

2.1.4 Slodowy slices

For a given nilpotent orbit \mathcal{O}_e , we will be interested in studying the structure of certain transverse slices to \mathcal{O}_e at a given point. While \mathcal{O}_e has many different transverse slices passing through any individual point, there is a certain relatively natural class of such slices which we'll be working with. Consider a nilpotent element $e \in \mathfrak{g}$ and complete it to an \mathfrak{sl}_2 -triple $\{e, h, f\}$ using the Jacobson–Morozov theorem.

Definition 2.1.15. The *Slodowy slice* to $\mathcal{O}_e \subseteq \mathfrak{g}$ through e is $\mathcal{S}_e := e + \mathfrak{z}(f)$, where $\mathfrak{z}(f)$ is the centraliser of f .

Remark 2.1.16. Though the varieties \mathcal{O}_e and \mathcal{S}_e defined above lie in the semisimple Lie algebra \mathfrak{g} , we can instead view them in \mathfrak{g}^* using the duality $\kappa: \mathfrak{g} \xrightarrow{\sim} \mathfrak{g}^*$ induced by the Killing form. In fact, many of the constructions developed later are more naturally considered in \mathfrak{g}^* . Which ambient space we are considering will generally be clear from the context, but when clarity is required we shall consider $\chi = \langle e, \cdot \rangle \in \mathfrak{g}^*$, $\mathcal{O}_\chi = G \cdot \chi \subseteq \mathfrak{g}^*$ its orbit under the coadjoint action, and $\mathcal{S}_\chi := \kappa(e + \mathfrak{z}(f)) = \chi + \ker \text{ad}^* f$.

Proposition 2.1.17. The Slodowy slice \mathcal{S}_e has a contracting \mathbb{C}^\times -action which fixes e .

Proof. Consider an \mathfrak{sl}_2 -triple containing e : this exponentiates to an embedding $\iota: SL_2 \hookrightarrow G$. We can then choose a cocharacter $\gamma: \mathbb{C}^\times \rightarrow G$ by defining $\gamma(t) = \iota \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}$. Note that $\text{Ad}_{\gamma(t)} e = t^2 e$.

Consider the action of \mathbb{C}^\times on \mathcal{S}_e given by $t \cdot (e + x) := t^{-2} \text{Ad}_{\gamma(t)}(e + x) = e + t^{-2} \text{Ad}_{\gamma(t)} x$ for $x \in \mathfrak{z}(f)$. We can see from the second equality that this action fixes e , so it just remains to show that $\lim_{t \rightarrow \infty} t \cdot (e + x) = e$ for any $x \in \mathfrak{z}(f)$. This follows from version of property GG4 for the nilpotent f , which implies that $\text{Ad}_{\gamma(t)}$ acts on $\mathfrak{z}(f)$ by negative powers of t , and hence $t^{-2} \text{Ad}_{\gamma(t)}$ acts by strictly negative powers. \square

Note. In the course of the above proof, we showed that by completing any nilpotent element to an \mathfrak{sl}_2 -triple we can express $t^2 e = \text{Ad}_{\gamma(t)} e$ for some cocharacter γ . This means that non-zero scalar multiplication of a nilpotent element preserves G -orbits.

Proposition 2.1.18. The Slodowy slice \mathcal{S}_e is a transverse slice to \mathcal{O}_e at the point e , and in particular $T_e \mathcal{S}_e \oplus T_e \mathcal{O}_e = T_e \mathfrak{g}$. Furthermore, e is the unique point of intersection: $\mathcal{S}_e \cap \mathcal{O}_e = \{e\}$.

Proof. [CG, Proposition 3.7.15] To prove that \mathcal{S}_e is a transverse slice to \mathcal{O}_e , it suffices to prove that $T_e \mathcal{S}_e \oplus T_e \mathcal{O}_e = T_e \mathfrak{g}$; the stronger result in a neighbourhood of e follows automatically by exponentiation.

Note that $T_e \mathcal{S}_e = \mathfrak{z}(f)$ and $T_e \mathcal{O}_e = [\mathfrak{g}, e]$; that $\mathfrak{z}(f) \cap [\mathfrak{g}, e] = 0$ follows directly from \mathfrak{sl}_2 representation theory. To show that they together span $T_e \mathfrak{g} = \mathfrak{g}$, we count dimensions. Consider the decomposition of \mathfrak{g} into irreducible \mathfrak{sl}_2 -modules $\mathfrak{g} = \bigoplus_{j=1}^n V(\lambda_j)$. The centraliser $\mathfrak{z}(f)$ consists of precisely those elements of \mathfrak{g} which lie in the lowest weight spaces of the irreducible components, and $[\mathfrak{g}, e]$ consists of those elements which do not lie in the lowest weight

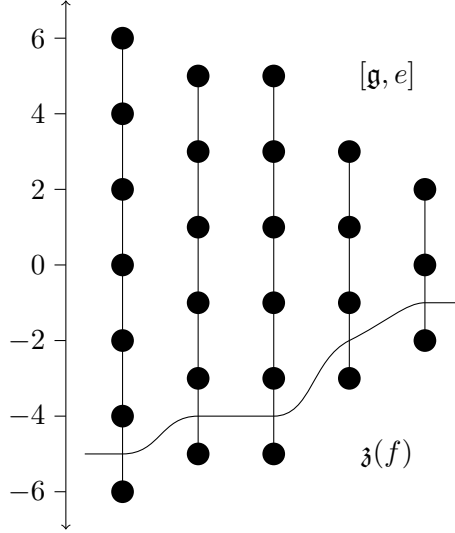


Figure 2.2: The decomposition of a finite-dimensional \mathfrak{sl}_2 -module as a sum of simple modules. The dimension of \mathfrak{g} is equal to the sum of the dimensions of $\mathfrak{z}(f)$ and $[\mathfrak{g}, e]$.

spaces of the irreducible components (see fig. 2.2). Thus $\dim \mathfrak{z}(f) + \dim [\mathfrak{g}, e] = \dim \mathfrak{g}$, and so $T_e \mathcal{S}_e \oplus T_e \mathcal{O}_e = T_e \mathfrak{g}$.

It remains to prove that $\mathcal{S}_e \cap \mathcal{O}_e = \{e\}$. We proved above that \mathcal{S}_e is a transverse slice to \mathcal{O}_e in some sufficiently small neighbourhood, so any other points of intersection must lie outside of that neighbourhood. We note that the contracting action of proposition 2.1.17 preserves G -orbits, as it is composed of an honest adjoint action of G followed by a scaling of the resulting nilpotent, which can also be expressed as a G -action by the note following the proposition. Thus we can contract any point of $(\mathcal{S}_e \cap \mathcal{O}_e) \setminus \{e\}$ to another point in the same set in an arbitrarily small neighbourhood of e , contradicting transversality of \mathcal{S}_e . \square

2.2 W-algebra basics

We now turn to defining the W-algebras themselves and establishing their basic properties. Throughout this section we shall develop a procedure for defining the W-algebra $U(\mathfrak{g}, e)$ given a nilpotent e and good grading Γ , establish its identity as a non-commutative filtered algebra, and remark on some of the geometry linking the W-algebras to Slodowy slices. In the process we shall prove the independence of the isomorphism class of the W-algebra $U(\mathfrak{g}, e)$ from the various choices our definition entails.

2.2.1 Premet subalgebras

For our definition of W-algebras, we need to introduce a class of nilpotent subalgebras compatible with the nilpotent element e . These subalgebras are known as *Premet subalgebras*, and they are almost entirely determined by a choice of a good grading for e . In the case of an even good grading we can unambiguously define a Premet subalgebra purely from the good grading itself, but if the grading has non-zero odd component we need to be a bit more subtle.

Lemma 2.2.1. *Let $\mathfrak{g} = \bigoplus_{j \in \mathbb{Z}} \mathfrak{g}_j$ be a good grading for the nilpotent e . The space \mathfrak{g}_{-1} is a symplectic vector space with symplectic form $\omega(x, y) := \langle e, [x, y] \rangle$.*

Proof. That the form ω is antisymmetric follows from the antisymmetry of the Lie bracket. To prove that it is non-degenerate, we show that its radical $\text{rad } \omega := \{x \in \mathfrak{g}_{-1} : \omega(x, y) = 0 \ \forall y \in \mathfrak{g}_{-1}\}$, is zero. Let $x \in \text{rad } \omega$: since $0 = \omega(x, \cdot) = \langle e, [x, \cdot] \rangle = \langle [e, x], \cdot \rangle$, we check to see when this vanishes as an operator on \mathfrak{g}_{-1} . By properties [GG2](#) and [GG5](#), $[e, x]$ is a non-zero element of \mathfrak{g}_1 and $\langle \mathfrak{g}_1, \mathfrak{g}_j \rangle = 0$ unless $j = -1$, so $[e, x]$ is a non-zero element of the radical of the Killing form. However, \mathfrak{g} is a semisimple Lie algebra and so its Killing form is non-degenerate. Therefore $x = 0$, $\text{rad } \omega = \{0\}$, and ω is non-degenerate. \square

Definition 2.2.2. Let e be a nilpotent element in \mathfrak{g} . A *Premet subalgebra* for e is a subalgebra $\mathfrak{m} \subseteq \mathfrak{g}$ constructed using a choice of a good grading $\Gamma: \mathfrak{g} = \bigoplus_{j \in \mathbb{Z}} \mathfrak{g}_j$ along with a Lagrangian subspace $\mathfrak{l} \subseteq \mathfrak{g}_{-1}$; it is defined as $\mathfrak{m} = \mathfrak{l} \oplus \bigoplus_{j \leq -2} \mathfrak{g}_j$.

Note that this is closed under the Lie bracket by the fact that Γ is a Lie algebra grading. Premet subalgebras are closely tied to the structure of the nilpotent orbit \mathcal{O}_e , and enjoy a number of useful properties.

Proposition 2.2.3. Let $\mathfrak{m} \subseteq \mathfrak{g}$ be a Premet subalgebra for a nilpotent e ; then:

1. $\dim \mathfrak{m} = \frac{1}{2} \dim \mathcal{O}_e$.
2. \mathfrak{m} is an *ad-nilpotent*, and in particular nilpotent, subalgebra of \mathfrak{g} .
3. the linear functional $\chi = \langle e, \cdot \rangle$ restricts to a character on \mathfrak{m} .

Proof. Property 1 follows from the orbit–stabiliser theorem and property [GG6](#):

$$\begin{aligned} \dim \mathcal{O}_e &= \dim \mathfrak{g} - \dim \mathfrak{z}(e) = \sum_{j \in \mathbb{Z}} \dim \mathfrak{g}_j - \dim \mathfrak{g}_0 - \dim \mathfrak{g}_1 = \\ &= \dim \mathfrak{g}_{-1} + \sum_{j \leq -2} \dim(\mathfrak{g}_j + \mathfrak{g}_{-j}) = \dim \mathfrak{g}_{-1} + 2 \sum_{j \leq -2} \dim \mathfrak{g}_j = 2 \dim \mathfrak{m}. \end{aligned}$$

Property 2 follows because $\mathfrak{m} \subseteq \bigoplus_{j \leq -1} \mathfrak{g}_j$, and so consists entirely of ad-nilpotent elements. For property 3 we note first that $e \in \mathfrak{g}_2$, and so property [GG5](#) implies that χ vanishes except on \mathfrak{g}_{-2} . Thus we know that $\chi([x, y]) = 0$ unless x and y both lie in $\mathfrak{l} = \mathfrak{m} \cap \mathfrak{g}_{-1}$; but $\chi([x, y]) = \omega(x, y)$, which vanishes since \mathfrak{l} was chosen to be Lagrangian with respect to ω . \square

Premet subalgebras for even good gradings

Fortunately the situation is simpler for even good gradings, and we can give an intrinsic characterisation of all Premet subalgebras which can be constructed from an even good grading. We note that if Γ is an even good grading, the Premet subalgebra $\mathfrak{m} = \bigoplus_{j < 0} \mathfrak{g}_j$ is the nilradical of a parabolic subalgebra $\mathfrak{p}^- = \bigoplus_{j \leq 0} \mathfrak{g}_j$ (cf. [\[CM, Lemma 3.8.4\]](#)). The converse is true with one additional condition. Choose a Cartan subalgebra \mathfrak{h} and set of simple roots Φ such that $\mathfrak{p}^- = \mathfrak{p}_{-\Theta}$ for some subset $\Theta \subseteq \Phi$. Let Δ^+ be the positive roots and Δ_{Θ}^+ the elements of Δ^+ with exactly one simple summand in Θ .

Theorem 2.2.4 (Elashvili–Kac). [\[EK, Theorem 2.1\]](#) If \mathfrak{m} is the nilradical of a parabolic subalgebra \mathfrak{p} in \mathfrak{g} , then it is a Premet subalgebra if and only if there exists a Richardson element – that is, an element of the open dense orbit in \mathfrak{m} under the adjoint action of P – lying in the subspace generated by $\{e_{\alpha} : \alpha \in \Delta_{\Theta}^+\}$.

Note. Though Richardson elements exist for any parabolic subalgebra, it is not always true that there exists one in satisfying the condition of the theorem. Such parabolics are called *nice parabolic subalgebras*, and their classification is equivalent to the classification of good gradings.

2.2.2 The Whittaker definition of W-algebras

We can now present a definition of the W-algebra $U(\mathfrak{g}, e)$ associated to the nilpotent e using a Premet subalgebra. Proposition 2.2.3 states that $\chi = \langle e, \cdot \rangle : \mathfrak{m} \rightarrow \mathbb{C}$ is a Lie algebra character, and so defines a one-dimensional representation \mathbb{C}_χ . We can induce this representation of $U(\mathfrak{m})$ to a representation of $U(\mathfrak{g})$, the result being known as the *generalised Gel'fand–Graev module* Q_χ :

$$Q_\chi := U(\mathfrak{g}) \otimes_{U(\mathfrak{m})} \mathbb{C}_\chi$$

Note that the kernel of the induced morphism $\chi : U(\mathfrak{m}) \rightarrow \mathbb{C}$ is generated as a two-sided ideal in $U(\mathfrak{m})$ by the shifted Lie algebra $\mathfrak{m}_\chi = \{a - \chi(a) : a \in \mathfrak{m}\}$. Considering the left ideal $U(\mathfrak{g})\mathfrak{m}_\chi$ in $U(\mathfrak{g})$, we can see that $Q_\chi \simeq U(\mathfrak{g})/U(\mathfrak{g})\mathfrak{m}_\chi$.

Proposition 2.2.5. *The Lie algebra \mathfrak{m}_χ acts locally nilpotently on Q_χ .*

Proof. Since Q_χ is a quotient of $U(\mathfrak{g})$, it suffices to see how \mathfrak{m}_χ acts on PBW monomials in $U(\mathfrak{g})$. Let $\bar{u} \in Q_\chi$ be such that $u = u_1 u_2 \cdots u_n$ is a PBW monomial. Since \mathfrak{m} consists of ad-nilpotent elements, for each $a \in \mathfrak{m}$ we can consider $N \in \mathbb{N}$ large enough so that $(\text{ad } a)^N(u_j) = 0$ for all j . Then $(\text{ad } a)^{nN}(u) = 0$, and hence large enough powers of a will commute with u ; it follows that $a - \chi(a)$ acts locally nilpotently. \square

Definition 2.2.6. Given a nilpotent $e \in \mathfrak{g}$, good grading Γ and Langrangian subspace $\mathfrak{l} \subseteq \mathfrak{g}_{-1}$, the (finite) W-algebra $U(\mathfrak{g}, e)$ is the space of Whittaker vectors in Q_χ , that is:

$$U(\mathfrak{g}, e) := (Q_\chi)^{\mathfrak{m}_\chi} = \{\bar{u} \in U(\mathfrak{g})/U(\mathfrak{g})\mathfrak{m}_\chi : (a - \chi(a))u \in U(\mathfrak{g})\mathfrak{m}_\chi \ \forall a \in \mathfrak{m}_\chi\}. \quad (2.1)$$

Remark 2.2.7. In fact, the identity $(a - \chi(a))u = [a, u] + u(a - \chi(a))$ allows us to further identify

$$U(\mathfrak{g}, e) = (Q_\chi)^{\text{ad } \mathfrak{m}} = \{\bar{u} \in U(\mathfrak{g})/U(\mathfrak{g})\mathfrak{m}_\chi : [a, u] \in U(\mathfrak{g})\mathfrak{m}_\chi \ \forall a \in \mathfrak{m}_\chi\}. \quad (2.2)$$

We can observe that $U(\mathfrak{g}, e)$ is not just a vector space, but also inherits an algebra structure from $U(\mathfrak{g})$, as

$$[a, u_1 u_2] = [a, u_1] u_2 + u_1 [a, u_2] \in U(\mathfrak{g})\mathfrak{m}_\chi u_2 + U(\mathfrak{g})\mathfrak{m}_\chi = U(\mathfrak{g})\mathfrak{m}_\chi,$$

where the final equality follows from eq. (2.1). This is due solely from the fact that χ is a character of \mathfrak{m} ; the quotient of a non-commutative algebra by a left ideal does not inherit an algebra structure in general.

There is another way to see the algebra structure on $U(\mathfrak{g}, e)$ by studying the structure of the endomorphisms of the Gel'fand–Graev module Q_χ . It is a cyclic module, so to specify an endomorphism it suffices to define the image of $\bar{1} \in Q_\chi$: as a vector space,

$$\text{End}_{U(\mathfrak{g})}(Q_\chi) = \text{Hom}_{U(\mathfrak{g})}(U(\mathfrak{g})/U(\mathfrak{g})\mathfrak{m}_\chi, Q_\chi) = (Q_\chi)^{\mathfrak{m}_\chi}.$$

This isomorphism as vector spaces is furthermore an isomorphism of algebras

$$U(\mathfrak{g}, e) \simeq \text{End}_{U(\mathfrak{g})}(Q_\chi)^{\text{op}}. \quad (2.3)$$

This can be taken as an alternate definition of the W-algebra.

Remark 2.2.8. It should be noted that these definitions don't just depend on the nilpotent e , but depend also on the choice of good grading Γ and Lagrangian subspace \mathfrak{l} . Fortunately, the resulting algebras are isomorphic for different choices of Γ and \mathfrak{l} . We will therefore omit them from the notation. The independence of Lagrangian subspace \mathfrak{l} will be shown in section 2.2.4, while the independence of Γ is shown by Brundan and Goodwin in [BG].

Example 2.2.9. If we restrict to the case where $e = 0$, the only good grading for e is the trivial grading $\mathfrak{g} = \mathfrak{g}_0$. We therefore have that $\chi = 0$, $\mathfrak{m}_\chi = \{0\}$, $Q_\chi = U(\mathfrak{g})$, and so $U(\mathfrak{g}, 0) = U(\mathfrak{g})$.

W-algebras for even good gradings

The above definition can be significantly simplified in the case that the good grading Γ is even, as the Lagrangian subspace \mathfrak{l} will vanish. Consider the parabolic subalgebra $\mathfrak{p} := \bigoplus_{j \geq 0} \mathfrak{g}_j$ and its opposite nilradical $\mathfrak{m} = \bigoplus_{j \leq -2} \mathfrak{g}_j$; we have a vector space decomposition $\mathfrak{g} \simeq \mathfrak{p} \oplus \mathfrak{m}$. The PBW theorem then implies that we have an algebra decomposition $U(\mathfrak{g}) = U(\mathfrak{p}) \oplus U(\mathfrak{g})\mathfrak{m}_\chi$. This provides us with a \mathfrak{m}_χ -module isomorphism $U(\mathfrak{g})/U(\mathfrak{g})\mathfrak{m}_\chi \simeq U(\mathfrak{p})$, where \mathfrak{m}_χ acts by the χ -twisted adjoint action. With these observations, eq. (2.2) reduces to

$$U(\mathfrak{g}, e) = U(\mathfrak{p})^{\text{ad } \mathfrak{m}} := \{u \in U(\mathfrak{p}) : [a, u] \in U(\mathfrak{g})\mathfrak{m}_\chi \ \forall a \in \mathfrak{m}\}. \quad (2.4)$$

As a result, $U(\mathfrak{g}, e)$ can be identified as a subalgebra of $U(\mathfrak{p})$, rather than a subquotient of $U(\mathfrak{g})$: this greatly simplifies calculations in many examples.

Example 2.2.10. A classical result of Kostant [Kos] states that for a regular nilpotent element $e_{\text{reg}} \in \mathfrak{g}$, $U(\mathfrak{g}, e_{\text{reg}}) \simeq Z(\mathfrak{g})$. We can demonstrate this here in an example.

Let $\mathfrak{g} = \mathfrak{sl}_2$, $e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, and $f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$; this implies that $\mathfrak{p} = \langle e, h \rangle$ and $\mathfrak{m}_\chi = \langle f-1 \rangle$. It can be checked that $\frac{1}{2}h^2 - h + 2e$ lies in $U(\mathfrak{p})^{\text{ad } \mathfrak{m}}$, and in fact it freely generates it. We therefore have that $U(\mathfrak{g}, e) = \mathbb{C}[\frac{1}{2}h^2 - h + 2e]$, which is isomorphic to $Z(\mathfrak{g}) = \mathbb{C}[\frac{1}{2}h^2 + ef + fe]$ under the projection $U(\mathfrak{g}) = U(\mathfrak{p}) \otimes U(\mathfrak{m}) \twoheadrightarrow U(\mathfrak{p})$.

2.2.3 The Kazhdan filtration

As it stands the W-algebra is an algebraic construction, however it is closely related to the geometry of Slodowy slices. In order to discuss this, we need to introduce a filtration on $U(\mathfrak{g}, e)$.

Recall that $U(\mathfrak{g})$ has a filtration known as the PBW filtration, where $U_j(\mathfrak{g})$ is spanned by the collection of all monomials $x_1 x_2 \cdots x_i$ for $i \leq j$, $x_k \in \mathfrak{g}$. We would like to modify this filtration to take the good grading Γ into account; consider a semisimple element h_Γ from lemma 2.1.3. We can extend the grading on \mathfrak{g} to a grading on $U(\mathfrak{g})$ by defining $U(\mathfrak{g})_i := \{u \in U(\mathfrak{g}) : \text{ad } h_\Gamma(u) = iu\}$, and combine the two by defining

$$U_j(\mathfrak{g})_i := U_j(\mathfrak{g}) \cap U(\mathfrak{g})_i = \{u \in U_j(\mathfrak{g}) : \text{ad } h_\Gamma(u) = iu\}.$$

Definition 2.2.11. The *Kazhdan filtration* on $U(\mathfrak{g})$ associated to the grading Γ of \mathfrak{g} is defined by

$$F_n U(\mathfrak{g}) = \sum_{i+2j \leq n} U_j(\mathfrak{g})_i.$$

Remark 2.2.12. The Kazhdan filtration enjoys a number of useful properties:

1. If we consider $x \in \mathfrak{g}_i$ and $y \in \mathfrak{g}_j$, it follows that $x \in F_{i+2}U(\mathfrak{g})$ and $y \in F_{j+2}U(\mathfrak{g})$, and therefore $[x, y] \in F_{i+j+2}U(\mathfrak{g})$. Hence the associated graded $\text{gr } U(\mathfrak{g})$ is commutative, and is thus isomorphic to $\text{Sym}(\mathfrak{g}) = \mathbb{C}[\mathfrak{g}^*]$.
2. The left ideal $U(\mathfrak{g})\mathfrak{m}_\chi$ contains all the elements of $U(\mathfrak{g})$ of strictly negative degree; therefore, the Kazhdan filtration descends to a positive filtration on the Gel'fand–Graev module $Q_\chi = U(\mathfrak{g})/U(\mathfrak{g})\mathfrak{m}_\chi$.
3. Passing to the associated graded algebra, the left ideal $U(\mathfrak{g})\mathfrak{m}_\chi$ passes to the two-sided ideal $\text{gr } U(\mathfrak{g})\mathfrak{m}_\chi \subseteq \mathbb{C}[\mathfrak{g}^*]$ consisting of all functions which vanish on $\chi + \mathfrak{m}^{*, \perp}$, where $\mathfrak{m}^{*, \perp} = \{\xi \in \mathfrak{g}^* : \xi(a) = 0 \ \forall a \in \mathfrak{m}\}$.

4. The associated graded of the Gel'fand–Graev module is $\text{gr } Q_\chi \simeq \text{gr } U(\mathfrak{g}) / \text{gr } U(\mathfrak{g})\mathfrak{m}_\chi$, which is a positively-graded commutative algebra in turn isomorphic to $\mathbb{C}[\chi + \mathfrak{m}^{*,\perp}]$. Furthermore, the natural map $\text{gr } Q_\chi \rightarrow \mathbb{C}[\chi + \mathfrak{m}^{*,\perp}]$ is an algebra isomorphism.
5. The Kazhdan filtration descends to a positive filtration of the W-algebra $U(\mathfrak{g}, e) = (Q_\chi)^{\mathfrak{m}_\chi}$, and furthermore $F_0 U(\mathfrak{g}, e) = \mathbb{C}$.

2.2.4 A geometric interpretation of W-algebras

These properties suggest a connection between the filtered algebras and Gel'fand–Graev module associated to the W-algebra and the functions on a number of subvarieties of \mathfrak{g}^* . The question then remains: what is the associated graded algebra of the W-algebra $U(\mathfrak{g}, e)$? We shall show that it corresponds to the ring of functions on the Slodowy slice $\mathbb{C}[\mathcal{S}_\chi]$ (recall section 2.1.4), however it will take a little effort to do so. The relationship between the associated graded algebras can be summarised in the following theorem. Recall the definition of a Γ -graded \mathfrak{sl}_2 -triple (definition 2.1.9).

Theorem 2.2.13 (Gan–Ginzburg). [GG] *Consider a nilpotent $e \in \mathfrak{g}$ with an associated good grading Γ and Premet subalgebra \mathfrak{m} . Let $\{e, h, f\}$ be a Γ -graded \mathfrak{sl}_2 -triple, and $\mathcal{S}_e \subseteq \mathfrak{g}$ and $\mathcal{S}_\chi \subseteq \mathfrak{g}^*$ the corresponding Slodowy slices. Then the following diagram of commutative algebras commutes:*

$$\begin{array}{ccccc}
 \text{gr } U(\mathfrak{g}) & \xlongequal{\quad} & \mathbb{C}[\mathfrak{g}^*] & \xleftarrow{\sim} & \mathbb{C}[\mathfrak{g}] \\
 \downarrow & & \downarrow & & \downarrow \\
 \text{gr } Q_\chi & \xlongequal{\quad} & \mathbb{C}[\chi + \mathfrak{m}^{*,\perp}] & \xleftarrow{\sim} & \mathbb{C}[e + \mathfrak{m}^\perp] \\
 \uparrow & & \downarrow & & \downarrow \\
 \text{gr } U(\mathfrak{g}, e) & \xrightarrow{\sim} & \mathbb{C}[\mathcal{S}_\chi] & \xleftarrow{\sim} & \mathbb{C}[\mathcal{S}_e]
 \end{array}$$

We first summarise the arrows we already know.

- The maps in the first column arise from the fact that $Q_\chi := U(\mathfrak{g}) / U(\mathfrak{g})\mathfrak{m}_\chi$ is a quotient module and $U(\mathfrak{g}, e) := (Q_\chi)^{\mathfrak{m}_\chi}$ is a submodule.
- The equalities between the first and second columns follow from remark 2.2.12.
- The isomorphisms between the second and third columns are induced by the isomorphism $\kappa: \mathfrak{g} \xrightarrow{\sim} \mathfrak{g}^*$ coming from the Killing form.
- The maps from the first to the second row in the second and third columns are both restriction of functions.

There are three arrows left to define: the bottom maps in the second and third columns, and the first map in the third row. However, it suffices to define the map $\mathbb{C}[e + \mathfrak{m}^\perp] \rightarrow \mathbb{C}[\mathcal{S}_e]$, and the remaining two follow:

- The map $\mathbb{C}[\chi + \mathfrak{m}^{*,\perp}] \rightarrow \mathbb{C}[\mathcal{S}_\chi]$ shall be defined by passing through the isomorphism κ .
- The final map $\text{gr } U(\mathfrak{g}, e) \rightarrow \mathbb{C}[\mathcal{S}_\chi]$ shall be defined as the composition of the arrows $\text{gr } U(\mathfrak{g}, e) \rightarrow \text{gr } Q_\chi \rightarrow \mathbb{C}[\chi + \mathfrak{m}^{*,\perp}] \rightarrow \mathbb{C}[\mathcal{S}_\chi]$.

Therefore, after defining the map $\mathbb{C}[e + \mathfrak{m}^\perp] \rightarrow \mathbb{C}[\mathcal{S}_e]$, to complete the proof of theorem 2.2.13 it only remains to show that the map $\text{gr } U(\mathfrak{g}, e) \rightarrow \mathbb{C}[\mathcal{S}_\chi]$ is an isomorphism. Our proof shall follow the exposition of Wang [Wan].

Remark 2.2.14. We will temporarily relax some of our definitions slightly to allow a bit more flexibility than strictly necessary to explain theorem 2.2.13. This does not complicate the proof, and it allows us to prove the independence of choice of Lagrangian subspace $\mathfrak{l} \subseteq \mathfrak{g}_{-1}$ in the definition of $U(\mathfrak{g}, e)$ as a side benefit.

Let \mathfrak{l} be an *isotropic subspace* of \mathfrak{g}_{-1} (i.e. $\omega(\mathfrak{l}, \mathfrak{l}) = 0$), rather than the stronger condition of being a Lagrangian subspace; we continue to define the Premet subalgebra $\mathfrak{m} := \mathfrak{l} \oplus \bigoplus_{j \leq -2} \mathfrak{g}_j$. We introduce the new definitions:

$$\mathfrak{l}' := \mathfrak{l}^{\perp\omega} = \{x \in \mathfrak{g}_{-1} : \omega(x, \mathfrak{l}) = 0\}, \quad (2.5)$$

$$\mathfrak{m}' := \mathfrak{l}' \oplus \bigoplus_{j \leq -2} \mathfrak{g}_j. \quad (2.6)$$

Note that if we choose \mathfrak{l} to be a Lagrangian (and hence isotropic) subspace, then it follows from the definition that $\mathfrak{l} = \mathfrak{l}'$ and $\mathfrak{m} = \mathfrak{m}'$, so everything we say for these spaces applies equally well with our original definitions.

We continue to define $Q_\chi := U(\mathfrak{g})/U(\mathfrak{g})\mathfrak{m}_\chi$, however we may now denote it $Q_\mathfrak{l}$ to emphasise the dependence on the choice of \mathfrak{l} . Finally, we define

$$U(\mathfrak{g}, e)_\mathfrak{l} := (Q_\mathfrak{l})^{\text{ad } \mathfrak{m}'}. \quad (2.7)$$

This retains an algebra structure by the same argument as for $U(\mathfrak{g}, e)$, and the Kazhdan filtration is also defined for $Q_\mathfrak{l}$ and $U(\mathfrak{g}, e)_\mathfrak{l}$. Further, all the maps of theorem 2.2.13 exist and satisfy the same properties when substituting our new definitions \mathfrak{m} , $Q_\mathfrak{l}$ and $U(\mathfrak{g}, e)_\mathfrak{l}$, *mutatis mutandis*.

Lemma 2.2.15. *For any Γ -graded \mathfrak{sl}_2 -triple $\{e, h, f\}$, it follows that $\mathfrak{m}^\perp = [\mathfrak{m}', e] \oplus \mathfrak{z}(f)$.*

Proof. We note first that $\mathfrak{z}(f) \subseteq \mathfrak{m}^\perp$, as $\mathfrak{m}^\perp \supseteq \bigoplus_{j \leq 0} \mathfrak{g}_j \supseteq \mathfrak{z}(f)$, where the first inclusion follows from property GG5 and the second is a version of property GG4 for f . We also have that $[\mathfrak{m}', e] \subseteq \mathfrak{m}^\perp$, as $\langle \mathfrak{m}, [\mathfrak{m}', e] \rangle = \langle [\mathfrak{m}, \mathfrak{m}'], e \rangle = 0$. In addition, $[\mathfrak{m}', e] \cap \mathfrak{z}(f) = \{0\}$, which follows from \mathfrak{sl}_2 representation theory (see fig. 2.1). Finally, note that $\dim \mathfrak{m}^\perp \cap \mathfrak{g}_1 = \dim \mathfrak{l}'$, and so $\dim \mathfrak{m}^\perp = \dim \mathfrak{m}' + \dim \mathfrak{g}_0 + \dim \mathfrak{g}_{-1}$; then $\dim \mathfrak{m}' = \dim [\mathfrak{m}', e]$ by property GG2, and $\dim \mathfrak{g}_0 + \dim \mathfrak{g}_{-1} = \dim \mathfrak{z}(f)$ by the version of property GG6 for f . \square

Remark 2.2.16. Note that this lemma shows that $\mathcal{S}_e = e + \mathfrak{z}(f)$ is a subvariety of $e + \mathfrak{m}^\perp$, and hence the remaining map $\mathbb{C}[e + \mathfrak{m}^\perp] \rightarrow \mathbb{C}[\mathcal{S}_e]$ in theorem 2.2.13 can be defined as restriction of functions. Theorem 2.2.13 therefore follows from the following theorem.

Theorem 2.2.17. *The map $\nu: \text{gr } U(\mathfrak{g}, e)_\mathfrak{l} \rightarrow \mathbb{C}[\mathcal{S}_\chi]$, defined as the composition $\text{gr } U(\mathfrak{g}, e)_\mathfrak{l} \rightarrow \text{gr } Q_\mathfrak{l} \rightarrow \mathbb{C}[e + \mathfrak{m}^{*, \perp}] \rightarrow \mathbb{C}[\mathcal{S}_\chi]$, is an isomorphism.*

Proof of theorem 2.2.17

Recall from proposition 2.1.17 the contracting \mathbb{C}^\times -action on \mathcal{S}_e , denoted $\rho(t) := t^{-2} \text{Ad}_{\gamma(t)}$. We note that this acts not just on \mathcal{S}_e but also on the whole of \mathfrak{g} , and that furthermore it stabilises not only $\mathfrak{z}(f)$ but also \mathfrak{m}^\perp . In addition, the action of \mathbb{C}^\times on $e + \mathfrak{m}^\perp$ is contracting for the same reasons as in proposition 2.1.17.

Let M' be the closed subgroup of G such that $\text{Lie } M' = \mathfrak{m}'$. We define a \mathbb{C}^\times -action on the variety $M' \times \mathcal{S}_e = M' \times (e + \mathfrak{z}(f))$ by the equation

$$t \cdot (g, e + x) = (\gamma(t)g\gamma(t^{-1}), \rho(t)(e + x)). \quad (2.8)$$

Note that the action is still a contracting action, and in particular

$$\lim_{t \rightarrow \infty} t \cdot (g, e + x) = (1, e). \quad (2.9)$$

Lemma 2.2.18. *The adjoint action map $\alpha: M' \times \mathcal{S}_e \rightarrow e + \mathfrak{m}^\perp$ is a \mathbb{C}^\times -equivariant isomorphism of affine varieties.*

Proof. The proof that the map is \mathbb{C}^\times -equivariant is a direct computation. Note that the map induces an isomorphism $T_{(1,e)}(M' \times \mathcal{S}_e) \xrightarrow{\sim} T_e(e + \mathfrak{m}^\perp)$ by lemma 2.2.15. The lemma then follows from the following general result: any \mathbb{C}^\times -equivariant map of smooth affine varieties with contracting \mathbb{C}^\times -actions which induces an isomorphism on tangent spaces at the fixed points must be an isomorphism. \square

To complete the proof, we note that $U(\mathfrak{g})$ and Q_l are \mathfrak{m}' -modules via the adjoint action, and that the map $U(\mathfrak{g}) \rightarrow Q_l$ is an \mathfrak{m}' -module homomorphism. The adjoint \mathfrak{m}' -action preserves the Kazhdan filtration, and so the associated graded map $\text{gr } U(\mathfrak{g}) \rightarrow \text{gr } Q_l$ is also an \mathfrak{m}' -module homomorphism. Noting that $U(\mathfrak{g}, e)_l = (Q_l)^{\mathfrak{m}'} = H^0(\mathfrak{m}', Q_l)$, we can therefore reformulate theorem 2.2.17 in terms of Lie algebra cohomology.

Theorem 2.2.19. *The map $\nu: \text{gr } U(\mathfrak{g}, e)_l \rightarrow \mathbb{C}[\mathcal{S}_\chi]$ can be decomposed as*

$$\text{gr } H^0(\mathfrak{m}', Q_l) \xrightarrow{\nu_1} H^0(\mathfrak{m}', \text{gr } Q_l) \xrightarrow{\nu_2} \mathbb{C}[\mathcal{S}_\chi],$$

where ν_1 and ν_2 are isomorphisms. Furthermore, $H^i(\mathfrak{m}', Q_l) = H^i(\mathfrak{m}', \text{gr } Q_l) = 0$ for $i > 0$.

Proof. Transferring lemma 2.2.18 through the isomorphism $\kappa: \mathfrak{g} \xrightarrow{\sim} \mathfrak{g}^*$ allows us identify the \mathfrak{m}' -module isomorphisms

$$\text{gr } Q_l \simeq \mathbb{C}[\chi + \mathfrak{m}^{*,\perp}] \simeq \mathbb{C}[M'] \otimes \mathbb{C}[\mathcal{S}_\chi],$$

where the \mathfrak{m}' -action on $\mathbb{C}[M'] \otimes \mathbb{C}[\mathcal{S}_\chi]$ comes by identifying $\mathfrak{m}' = T_1 M'$ as the derivations of $\mathbb{C}[M']$ and acting on the first tensor component.

Recall the standard co-chain complex for calculating Lie algebra cohomology $H^i(\mathfrak{m}', X)$:

$$0 \rightarrow X \rightarrow \mathfrak{m}'^* \otimes X \rightarrow \bigwedge^2 \mathfrak{m}'^* \otimes X \rightarrow \cdots \rightarrow \bigwedge^k \mathfrak{m}'^* \otimes X \rightarrow \cdots \quad (2.10)$$

In the case that $X = \mathbb{C}[M']$ this is just the de Rham complex $\Omega^\bullet M'$, and so we've reduced to calculating the de Rham cohomology of M' . But since M' is a unipotent group and therefore an affine space, we simply have that $H^0(\mathfrak{m}', \mathbb{C}[M']) = \mathbb{C}$ and all higher cohomology groups vanish. It follows that

$$H^0(\mathfrak{m}', \text{gr } Q_l) \simeq H^0(\mathfrak{m}', \mathbb{C}[M']) \otimes \mathbb{C}[\mathcal{S}_\chi] = \mathbb{C}[\mathcal{S}_\chi],$$

and hence the map ν_2 is an isomorphism and all higher cohomology groups $H^k(\mathfrak{m}', \text{gr } Q_l)$ vanish for $k > 0$.

Remark 2.2.20. Note that we could further remark that

$$H^0(\mathfrak{m}', \text{gr } Q_{\mathfrak{l}}) \simeq H^0(\mathfrak{m}', \mathbb{C}[\chi + \mathfrak{m}^{*,\perp}]) = \mathbb{C}[\chi + \mathfrak{m}^{*,\perp}]^{M'},$$

that is to say the Slodowy slice is isomorphic to the quotient variety $\mathcal{S}_\chi \simeq (\chi + \mathfrak{m}^{*,\perp})/M'$. This interpretation will be very important for our future considerations.

We now define a filtration on the complex (2.10) for $X = Q_{\mathfrak{l}}$. Note that \mathfrak{m}' is a strictly-negatively graded subalgebra of \mathfrak{g} under the grading Γ , and so \mathfrak{m}'^* is strictly-positively graded; we write this grading as $\mathfrak{m}'^* = \bigoplus_{j \geq 1} \mathfrak{m}'^*_j$. We can combine this with the Kazhdan filtration on $Q_{\mathfrak{l}}$ to define the filtration $F_p(\bigwedge^k \mathfrak{m}'^* \otimes Q_{\mathfrak{l}})$ as spanned by all $(x_1 \wedge \cdots \wedge x_k) \otimes v$ for which $x_j \in \mathfrak{m}'^*_{i_j}$, $v \in F_n Q_{\mathfrak{l}}$ and $n + \sum_{j=1}^k i_j \leq p$. Taking the associated graded of this filtered complex gives us the standard complex for computing the cohomology of $\text{gr } Q_{\mathfrak{l}}$, as \mathfrak{m}'^* is already a graded Lie algebra.

Consider the spectral sequence with $E_0^{p,q} := F_p(\bigwedge^{p+q} \mathfrak{m}'^* \otimes Q_{\mathfrak{l}}) / F_{p-1}(\bigwedge^{p+q} \mathfrak{m}'^* \otimes Q_{\mathfrak{l}})$. We note that the first page of this spectral sequence is just $E_1^{p,q} = H^{p+q}(\mathfrak{m}', \text{gr}_p Q_{\mathfrak{l}})$, and since this is zero for $p + q \neq 0$, the spectral sequence degenerates at the first page. Hence the spectral sequence will converge to $H^{p+q}(\mathfrak{m}', \text{gr}_p Q_{\mathfrak{l}})$. However, by general results this spectral sequence, if it converges, must converge to $E_\infty^{p,q} = F_p H^{p+q}(\mathfrak{m}', Q_{\mathfrak{l}}) / F_{p-1} H^{p+q}(\mathfrak{m}', Q_{\mathfrak{l}})$ (see a standard reference, e.g. [McC]). Hence it must be that $H^i(\mathfrak{m}', \text{gr } Q_{\mathfrak{l}}) \simeq \text{gr } H^i(\mathfrak{m}', Q_{\mathfrak{l}})$, completing the proof of theorem 2.2.19, and therefore proving theorems 2.2.13 and 2.2.17. \square

Corollary 2.2.21. *The W-algebra $U(\mathfrak{g}, e)$ does not depend up to isomorphism on the choice of Lagrangian (or isotropic) subspace \mathfrak{l} .*

Proof. Let $\mathfrak{l} \subseteq \mathfrak{l}'$ be two isotropic subspaces of \mathfrak{g}_{-1} . The inclusion of \mathfrak{l} into \mathfrak{l}' induces a map $Q_{\mathfrak{l}} \rightarrow Q_{\mathfrak{l}'}$, which itself induces a map $U(\mathfrak{g}, e)_{\mathfrak{l}} \rightarrow U(\mathfrak{g}, e)_{\mathfrak{l}'}$. By theorem 2.2.17 this map descends to an isomorphism at the associated graded level, and is therefore an isomorphism itself. Choosing $\mathfrak{l} = \{0\}$ we get an isomorphism $U(\mathfrak{g}, e)_{\mathfrak{l}} \xrightarrow{\sim} U(\mathfrak{g}, e)_{\mathfrak{l}'}$ for any isotropic subspace \mathfrak{l}' (and therefore any Lagrangian subspace). \square

2.3 Hamiltonian reduction and its relation to W-algebras

The observation made in remark 2.2.20 gives us an important insight into the geometry of the Slodowy slice \mathcal{S}_χ : it can be expressed as a quotient of a certain affine subvariety of \mathfrak{g}^* . This recalls the construction of Hamiltonian reduction, where a Poisson variety with a Hamiltonian group action can be reduced at a regular value of the moment map. In fact, both \mathfrak{g}^* and \mathcal{S}_χ have the structure of a Poisson variety, and we can describe a Hamiltonian group action on \mathfrak{g}^* so that \mathcal{S}_χ is the Hamiltonian reduction by this action. To develop this point of view, we need to describe the Poisson structures on \mathfrak{g}^* and \mathcal{S}_χ .

2.3.1 The Slodowy slice as a Poisson variety

For any Lie algebra \mathfrak{g} , the dual space \mathfrak{g}^* has the natural structure of a Poisson variety coming from the Lie bracket on \mathfrak{g} . To define this, we first note that for any function $f \in \mathbb{C}[\mathfrak{g}^*]$, its differential $df \in T^*(\mathfrak{g}^*)$ can be viewed at any point $\xi \in \mathfrak{g}^*$ as lying naturally in the Lie algebra \mathfrak{g} ; this comes from the natural identification $T_\xi^*(\mathfrak{g}^*) = (\mathfrak{g}^*)^* \simeq \mathfrak{g}$.

Definition 2.3.1. For a Lie algebra \mathfrak{g} , the ring of functions $\mathbb{C}[\mathfrak{g}^*]$ has a natural Poisson bracket, called the *Lie–Poisson* or *Kostant–Kirillov* bracket. Given $f, g \in \mathbb{C}[\mathfrak{g}^*]$, the Poisson bracket is defined as $\{f, g\} \in \mathbb{C}[\mathfrak{g}^*]$, where $\{f, g\}(\xi) := \xi([df_\xi, dg_\xi])$.

Note. The Lie–Poisson bracket has an extremely simple expression. Any two elements $x, y \in \mathfrak{g}$ we can be interpreted as functions in $\mathbb{C}[\mathfrak{g}^*]$ using the evaluation map: the Poisson bracket $\{x, y\}$ is then just $[x, y]$. This can be extended to all polynomials in $\mathbb{C}[\mathfrak{g}^*]$ using the Leibniz rule.

Theorem 2.3.2. *The symplectic leaves of \mathfrak{g}^* with the Lie–Poisson bracket are the co-adjoint orbits.*

Proof. Choosing a co-adjoint orbit $\mathcal{O} \subseteq \mathfrak{g}^*$, we need to show that the Poisson bracket restricts to a symplectic form on \mathcal{O} . To do this, we only need to show that for each $\alpha \in \mathcal{O}$ the induced Poisson bracket on $T_\alpha \mathcal{O}$ is non-degenerate.

We note that for $x \in \mathfrak{g}$, viewed as an element of $\mathbb{C}[\mathfrak{g}^*]$,

$$\{x, \cdot\}(\xi) = \xi([x, \cdot]) = \xi(\text{ad}_x(\cdot)) = \text{ad}_x^*(\xi)(\cdot).$$

So at the point $\xi \in \mathfrak{g}^*$, the radical of the Poisson bracket is the set of all $x \in \mathfrak{g}$ which annihilate ξ under the co-adjoint action. This is the tangent space of G^ξ , the stabiliser of ξ in G . Since $\mathcal{O}_\xi \simeq G/G^\xi$ by the orbit–stabiliser theorem, the co-adjoint orbits are the maximal subvarieties on which the Poisson bracket is non-degenerate, and are hence the symplectic leaves. \square

Theorem 2.3.3. [GG] *The Slodowy slice \mathcal{S}_χ inherits a Poisson bracket from \mathfrak{g}^* .*

Proof. We recall from proposition 2.1.18 that the Slodowy slice \mathcal{S}_χ is transverse to \mathcal{O}_χ at χ . By standard results (cf. [Vai, Proposition 3.10]), it suffices to show that for any co-adjoint orbit \mathcal{O} and $\xi \in \mathcal{O} \cap \mathcal{S}_\chi$, the restriction of the symplectic form on $T_\xi \mathcal{O}$ to $T_\xi(\mathcal{O} \cap \mathcal{S}_\chi)$ is non-degenerate.

We will work in \mathfrak{g} , using the Killing isomorphism $\kappa: \mathfrak{g} \xrightarrow{\sim} \mathfrak{g}^*$ to pass to \mathfrak{g}^* when necessary. As $T_\xi(\mathcal{O} \cap \mathcal{S}_\chi) = T_\xi \mathcal{O} \cap T_\xi \mathcal{S}_\chi$, we will examine the spaces $T_\xi \mathcal{O}$ and $T_\xi \mathcal{S}_\chi$. Since \mathcal{O} is a co-adjoint orbit, its tangent space is simply the image of ad_ξ^* , which expressed in \mathfrak{g} is $[\kappa^{-1}(\xi), \mathfrak{g}]$, the image of $\text{ad}_{\kappa^{-1}(\xi)}$. The tangent space of $\mathcal{S}_\chi = \kappa(e + \mathfrak{z}(f))$ can be seen to be $\kappa(\mathfrak{z}(f))$.

The symplectic form at ξ can be seen to be $\omega_\xi(x, y) = \xi([x, y])$, so we need to determine the radical of ω_ξ restricted to $T_\xi(\mathcal{O} \cap \mathcal{S}_\chi)$. The annihilator of $T_\xi \mathcal{S}_\chi$ in \mathfrak{g} is $[f, \mathfrak{g}]$, which can be seen from associativity of the Killing form $\langle \mathfrak{z}(f), [f, \mathfrak{g}] \rangle = \langle \mathfrak{z}(f), f \rangle$, and so $\text{rad } \omega$ is

$$\kappa([\kappa^{-1}(\xi), [f, \mathfrak{g}]] \cap \mathfrak{z}(f))$$

Since $\kappa^{-1}(\xi) \in e + \mathfrak{z}(f)$, \mathfrak{sl}_2 representation theory tells us that this space is $\{0\}$ (see fig. 2.1). \square

2.3.2 Slodowy slices and Hamiltonian reduction

Since we now know that the space $X = \mathfrak{g}^*$ is a Poisson variety, we can ask whether the co-adjoint action of G on X is a Hamiltonian action. The answer is yes, and in order to demonstrate this we will exhibit a co-moment map $\mu^*: \mathfrak{g} \rightarrow \mathbb{C}[X]$, where $\mathfrak{g} = \text{Lie}(G)$. Note that this is equivalent to exhibiting a moment map $\mu: X \rightarrow \mathfrak{g}^*$ by the equation $\langle \mu(\xi), x \rangle = \mu^*(x)(\xi)$, where $x \in \mathfrak{g}$, $\xi \in X$ and $\langle \cdot, \cdot \rangle$ is the canonical pairing of \mathfrak{g}^* with \mathfrak{g} .

Note that G acts on X by the co-adjoint action Ad^* , which induces an action of \mathfrak{g} on X , the co-adjoint action ad^* . This in turn induces an action on $\mathbb{C}[X] = \text{Sym}(\mathfrak{g})$, which is just the adjoint action of \mathfrak{g} on $\text{Sym}(\mathfrak{g})$. To produce a co-moment map, we need a function $\mu^*: \mathfrak{g} \rightarrow \mathbb{C}[X]$ such that $\{\mu^*(x), \cdot\} = \text{ad}_x$; we claim that the map given by $\mu^*(x) = x$ suffices. To check this, note

that $\{\mu^*(x), \cdot\}(\alpha) = \langle [x, \cdot], \alpha \rangle = \langle \text{ad}_x(\cdot), \alpha \rangle$, and hence $\{\mu^*(x), \cdot\} = \text{ad}_x$. The corresponding moment map $\mu: \mathfrak{g}^* \rightarrow \mathfrak{g}^*$ is the identity map.

As a special case of the above, note that given a choice of Premet subalgebra \mathfrak{m} , its corresponding algebraic group $M \subseteq G$ acts on \mathfrak{g}^* by the co-adjoint action; the resulting moment map $\mu: \mathfrak{g}^* \rightarrow \mathfrak{m}^*$ is restriction of functions. Recall that in remark 2.2.20, we noted that $\mathcal{S}_\chi \simeq (\chi + \mathfrak{m}^{*,\perp})/M$. There are three important remarks to be made about this equation:

- that χ is a character of \mathfrak{m}^* implies that it is fixed under the co-adjoint action of M ;
- the space $\chi + \mathfrak{m}^{*,\perp}$ is the pre-image of χ under the moment map μ ; and
- the character χ is furthermore a regular value of the moment map μ , as $\mathfrak{m}^* \subseteq \mathfrak{m}^{*,\perp}$.

This proves the following theorem, which is fundamental to our further work.

Theorem 2.3.4. *The Slodowy slice $\mathcal{S}_\chi \subseteq \mathfrak{g}^*$ can be expressed as a Hamiltonian reduction of \mathfrak{g}^* at the regular value $\chi \in \mathfrak{m}^*$ by the co-adjoint action of M . Concretely,*

$$\mathcal{S}_\chi \simeq \mu^{-1}(\chi)/M \quad \text{and} \quad \mathbb{C}[\mathcal{S}_\chi] \simeq (\mathbb{C}[\mathfrak{g}^*]/I(\mu^{-1}(\chi)))^M.$$

As a Hamiltonian reduction of a Poisson variety, the Slodowy slice \mathcal{S}_χ inherits a Poisson bracket from \mathfrak{g}^* , defined as follows. Consider the following natural maps:

$$\begin{aligned} \iota: (\mathbb{C}[\mathfrak{g}^*]/I(\mu^{-1}(\chi)))^M &\hookrightarrow \mathbb{C}[\mathfrak{g}^*]/I(\mu^{-1}(\chi)) \\ \pi: \mathbb{C}[\mathfrak{g}^*] &\twoheadrightarrow \mathbb{C}[\mathfrak{g}^*]/I(\mu^{-1}(\chi)) \end{aligned}$$

Considering $f, g \in \mathbb{C}[\mathcal{S}_\chi]$, we define the Poisson bracket $\{f, g\}$ by lifting $\iota(f)$ and $\iota(g)$ to functions $\tilde{f}, \tilde{g} \in \mathbb{C}[\mathfrak{g}^*]$, and requiring that $\iota(\{f, g\}) = \pi(\{\tilde{f}, \tilde{g}\})$. That this is well-defined follows from the conditions of Hamiltonian reduction. This Poisson bracket agrees with that of theorem 2.3.3.

2.3.3 Quantum Hamiltonian reduction

Up to this point, we have been developing two related but distinct threads: Slodowy slices and W-algebras. In fact, these two subjects are much more intimately related than has been presented so far, and their correspondence forms the backbone of this thesis. In short, it can be stated that the W-algebra $U(\mathfrak{g}, e)$ is a quantisation of the ring of functions on the Slodowy slice \mathcal{S}_χ , and every statement about Slodowy slices has a corresponding quantisation which can be applied to W-algebras.

Recalling the Whittaker definition of W-algebras from eq. (2.2), the similarity to the Hamiltonian reduction equation of theorem 2.3.4 can be seen:

$$U(\mathfrak{g}, e) = (U(\mathfrak{g})/U(\mathfrak{g})\mathfrak{m}_\chi)^{\mathfrak{m}} \quad \mathbb{C}[\mathcal{S}_\chi] \simeq (\mathbb{C}[\mathfrak{g}^*]/I(\mu^{-1}(\chi)))^M$$

In the definition of the W-algebra, we consider the invariants under the adjoint action of \mathfrak{m} in the quotient of a non-commutative filtered algebra by a left ideal. This corresponds exactly in the Hamiltonian reduction expression to taking the invariants under the adjoint action of M in the quotient of the corresponding associated graded spaces.

This observation is more than a curiosity: there is a very precise sense in which we can translate concepts of Hamiltonian reduction of Poisson varieties to apply to the W-algebra $U(\mathfrak{g}, e)$.

This will allow us to express $U(\mathfrak{g}, e)$ as a *quantum Hamiltonian reduction* of the universal enveloping algebra $U(\mathfrak{g})$. This can be expressed using the formalism of *deformation quantisation*, which encodes the structure of a Poisson algebra in a non-commutative filtered algebra. Many of the structures of Poisson geometry have quantum analogues which can be applied to the W-algebraic context, including a quantum co-moment map which quantises the classical co-moment map. This furnishes us with all the tools necessary to define a quantum Hamiltonian reduction which descends gracefully to classical Hamiltonian reduction.

The technical details of this correspondence shall be addressed in detail in chapter 4, however for current purposes we can use this formalism as a motivation. This point of view will allow for the application of the techniques of Poisson geometry to W-algebras, providing us with powerful tools for solving problems.

Chapter 3

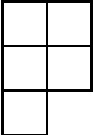
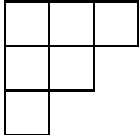
W-algebras in type A

Up to this point we've discussed the background for W-algebras in an arbitrary semisimple Lie algebra \mathfrak{g} . Much of what has been discussed has a much simpler and more concrete realisation in the *classical* Lie algebras, and in particular in the Lie algebras of type A. We shall discuss what this background looks like when we restrict our attention to the Lie algebras of type A.

In this chapter, we continue working over an algebraically closed field of characteristic zero, which can be taken to be \mathbb{C} . We shall fix a number $n \in \mathbb{N}_+$, and shall fix \mathfrak{g} to be a simple Lie algebra of type A_{n-1} . This is to say we shall let $\mathfrak{g} = \mathfrak{sl}_n$, the Lie algebra of $n \times n$ traceless matrices with the commutator Lie bracket $[x, y] := xy - yx$.

3.1 Nilpotent orbits

The nilpotent orbits in type A have a particularly simple characterisation. Since every matrix in $\text{Mat}_n(\mathbb{C})$ has a Jordan canonical form, every element of \mathfrak{g} can be conjugated to some matrix consisting of Jordan blocks on the diagonal. In particular, for a nilpotent element $e \in \mathfrak{g}$, every generalised eigenvalue is 0, so its conjugacy class is entirely determined by the sizes of the Jordan blocks. As a result, the nilpotent orbits in \mathfrak{g} are in bijection with the set of *partitions* of n , i.e. tuples $\lambda = (\lambda_1, \dots, \lambda_k)$ satisfying $\sum_{i=1}^k \lambda_i = n$ and $\lambda_1 \geq \dots \geq \lambda_k$. Partitions can be indicated by *Young diagrams*.

	$e = \begin{pmatrix} 0 & 1 & & \\ 0 & 0 & & \\ & & 0 & 1 \\ & & 0 & 0 \end{pmatrix}$	$e = \begin{pmatrix} 0 & 1 & 0 & & \\ 0 & 0 & 1 & & \\ 0 & 0 & 0 & & \\ & & & 0 & 1 \\ & & & 0 & 0 \end{pmatrix}$
Partition:	(2, 2, 1)	(3, 2, 1)
Young diagram:		

The set of nilpotent orbits has a natural partial ordering, where $\mathcal{O}' \leq \mathcal{O}$ if and only if $\mathcal{O}' \subseteq \overline{\mathcal{O}}$. Since nilpotent orbits correspond to partitions, this also imposes a partial ordering on the partitions of n . However, there already exists a well-known ordering on the set of partitions of n : the *dominance ordering*.

Definition 3.1.1. Let $\lambda = (\lambda_1, \dots, \lambda_k)$ and $\mu = (\mu_1, \dots, \mu_\ell)$ be two partitions of n satisfying $\lambda_1 \geq \dots \geq \lambda_k > 0$ and $\mu_1 \geq \dots \geq \mu_\ell > 0$. The *dominance ordering* is the partial ordering on the

set of partitions of n where $\lambda \geq \mu$ if and only if

$$\sum_{i=1}^j \lambda_i \geq \sum_{i=1}^j \mu_i$$

for every j between 1 and $\max(k, \ell)$ (here we declare $\lambda_i = \mu_j = 0$ for all $i > k$ and $j > \ell$). In this case, we say that λ *dominates* μ .

Theorem 3.1.2 (Gerstenhaber, Hesselink). [CM, Theorem 6.2.5] *The partial ordering on orbits corresponds to the dominance ordering under the equivalence between nilpotent orbits and partitions.*

Recall that in a partial ordering \geq , we say that λ *covers* μ if $\lambda > \mu$ and there exists no element ν such that $\lambda > \nu > \mu$, i.e. λ is a minimal element satisfying $\lambda > \mu$. We will need the following result on the fine structure of the dominance ordering.

Proposition 3.1.3 (Gerstenhaber). [CM, Lemma 6.2.4] *Let $\lambda = (\lambda_1, \dots, \lambda_k)$ and μ be two partitions of n . The partition λ covers μ if and only if μ can be obtained from λ by the following procedure. Let i be an index and $j > i$ be the smallest index such that $0 \leq \lambda_j < \lambda_i - 1$, where we again declare $\lambda_j = 0$ for $j > k$. Assume that either $\lambda_j = \lambda_i - 2$ or $\lambda_\ell = \lambda_i$ whenever $i < \ell < j$. Then the components of μ are obtained from the components of λ by replacing λ_i and λ_j by $\lambda_i - 1$ and $\lambda_j + 1$, respectively, and re-arranging if necessary.*

Example 3.1.4.

- (a) The partition $(3, 1)$ covers the partition $(2, 2)$.
- (b) The partition $(3, 3)$ covers the partition $(3, 2, 1)$.
- (c) The partition $(3, 2, 1)$ covers *both* the partitions $(2, 2, 2)$ and $(3, 1, 1, 1)$.
- (d) The partition $(4, 3, 1)$ *does not* cover the partition $(3, 3, 2)$. Taking $i = 1$ and $j = 3$ satisfies neither of the two conditions $\lambda_j = \lambda_i - 2$ and $\lambda_\ell = \lambda_i$ for all $i < \ell < j$. Instead, it can be seen that:
 - (i) The partition $(4, 3, 1)$ covers the partition $(4, 2, 2)$.
 - (ii) The partition $(4, 2, 2)$ covers the partition $(3, 3, 2)$.

Proof of proposition 3.1.3. By construction, λ will cover μ for any λ and μ related by the procedure in the proposition, so it remains to show the converse. Assume that λ covers μ ; let i be the least integer such that $\lambda_i > \mu_i$, and j be as in the proposition. Applying the procedure we obtain a new partition ν satisfying $\lambda > \nu \geq \mu$, and hence $\nu = \mu$ since λ covers μ . \square

Furthermore, recall from example 2.1.2 that there is an algorithm for constructing \mathfrak{sl}_2 -triples from a nilpotent e in type A. This not only allows us to obtain ordinary \mathfrak{sl}_2 -triples, but combining it with the characterisation of good gradings Γ from the following section will allow us to find Γ -graded \mathfrak{sl}_2 -triples as well. Hence all the concepts whose existence was proven in section 2.1 have concrete constructions in type A. This shall be used extensively within this chapter.

3.2 Pyramids

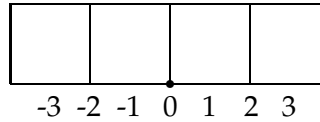
We are interested in determining what all possible good gradings are in type A. This has been accomplished by Elashvili and Kac, who have furthermore given a complete classification of all good gradings in every simple Lie algebra [EK]. In the classical types, this is accomplished using a combinatorial structure known as a *pyramid*, which is closely related to a Young diagram with additional information to encode the data of the grading. In type A there is a particularly simple description of the pyramids.

Definition 3.2.1. Let $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_k)$ be a partition of n , that is to say a sequence of strictly positive integers $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_k > 0$ where $\sum_{j=1}^k \lambda_j = n$. A *pyramid of shape λ* is a collection of n boxes of size 2×2 centred at integer points (i, j) of the plane, satisfying the following conditions:

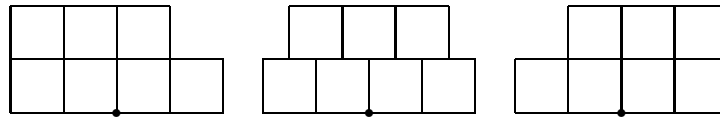
1. the number of boxes in the ℓ th row (which corresponds to having second co-ordinate $2\ell - 1$) is λ_ℓ for each $1 \leq \ell \leq k$.
2. the first co-ordinates of the boxes in the ℓ th row form an arithmetic sequence of difference 2, i.e. $f_\ell, f_\ell + 2, \dots, F_\ell$.
3. the first row is centred at 0, i.e. $f_1 = -F_1$.
4. the first co-ordinates of the first and last boxes in each row form increasing and decreasing sequences, respectively, i.e. $f_1 \leq \dots \leq f_k$ and $F_1 \geq \dots \geq F_k$.

More generally, a *pyramid of size n* is a pyramid of shape λ for some partition λ of n .

Remark 3.2.2. We shall illustrate this definition by constructing all possible pyramids of shape $\lambda = (4, 3)$. We begin by constructing a row of $\lambda_1 = 4$ boxes, each of width 2 and centred at integer values, such that the whole row is centred at zero.



We now place a second row of $\lambda_2 = 3$ boxes on top of the first row (again with each box centred on an integer) according to the following rule: the leftmost box of the second row cannot be further left than the leftmost box of the first row, and the rightmost box of the second row cannot be further right than the rightmost box of the first row. In our example, this gives us three pyramids:



If there were further λ_j in the partition, we would repeat this rule until all the rows were constructed.

Given a pyramid P with n boxes we can construct a *filling* of the pyramid by labelling each box with one of the numbers $\{1, \dots, n\}$ such that there are no repeated labels. Most often we shall choose the labelling so that it increases first up columns and then left to right.

3	6	
2	5	
1	4	7

(3.1)

Let P be a pyramid with a filling consisting of the numbers $\{1, \dots, n\}$. If the box labelled k is centred at the point $(i, 2j - 1)$, then we define:

- $\text{col}(k) = i$, the column number of the box labelled k .
- $\text{row}(k) = j$, the row number of the box labelled k .

Furthermore, we say ℓ is right-adjacent to k , denoted $k \rightarrow \ell$, if the box labelled ℓ lies in the same row as and immediately adjacent to the right of the box labelled k , i.e. $\text{row}(\ell) = \text{row}(k)$ and $\text{col}(\ell) = \text{col}(k) + 2$.

Example 3.2.3. In the above pyramid (3.1): $\text{row}(2) = 2$, $\text{col}(2) = -1$, $\text{row}(7) = 1$, $\text{col}(7) = 2$, while $1 \rightarrow 4$, $4 \rightarrow 7$, $2 \rightarrow 5$ and $3 \rightarrow 6$.

3.2.1 A bijection between pyramids and good gradings

To any filled pyramid P of size n , one can associate a nilpotent element e_P and a \mathbb{Z} -grading Γ_P by the following construction. Fix a standard basis of \mathbb{C}^n , and let E_{ij} be the matrix which maps the i th standard basis vector to the j th and maps all other basis vectors to zero. Note that this is the matrix with (j, i) -entry 1 and all other entries 0. Define the nilpotent e_P and the \mathbb{Z} -grading Γ_P by declaring

$$e_P := \sum_{i \rightarrow j} E_{ij} \quad \text{and} \quad \deg E_{ij} := \text{col}(j) - \text{col}(i).$$

One can check that the element e_P is nilpotent, as

$$e_P^k = \sum_{i_1 \rightarrow \dots \rightarrow i_k} E_{i_1 i_k},$$

which vanishes for large enough k as every row is of finite length. A straightforward calculation shows that Γ_P is a Lie algebra grading.

Note. The element e_P and the \mathbb{Z} -grading Γ_P can equally well be viewed in the context of the Lie algebra \mathfrak{sl}_n and the Lie algebra \mathfrak{gl}_n . In fact, the following proposition holds equally well for both \mathfrak{sl}_n and \mathfrak{gl}_n . The proofs in the two cases are virtually identical, so we shall proceed assuming the Lie algebra is \mathfrak{gl}_n for computational simplicity. The only substantive difference arises in determining the dimension of $\mathfrak{z}(e_P)$, which includes the centre of \mathfrak{gl}_n , but is of dimension one less when considering the centreless \mathfrak{sl}_n . Correcting for this discrepancy, the proof remains identical.

Proposition 3.2.4 (Elashvili–Kac). [EK, Theorem 4.1] *The grading Γ_P is good for e_P .*

Proof. It is clear by construction that $\deg e_P = 2$, so one only needs to show that $\text{ad } e_P: \mathfrak{g}_j \rightarrow \mathfrak{g}_{j+2}$ is injective for $j \leq -1$ by remark 2.1.6. Since $\ker \text{ad } e_P = \mathfrak{z}(e_P)$, to show that Γ_P is good for the nilpotent e_P , it suffices to prove the following claim.

Claim. Every element of $\mathfrak{g} = \mathfrak{gl}_n$ which commutes with e_P lies in the subspace $\bigoplus_{j \geq 0} \mathfrak{g}_j$.

It will be useful to express elements of \mathfrak{gl}_n , i.e. endomorphisms of \mathbb{C}^n , as arrow diagrams in the pyramid P . To the endomorphism E_{ij} we will assign the diagram consisting of the pyramid P with an arrow originating from the box labelled i and ending at the box labelled j . The diagram representing the nilpotent e_P is shown in fig. 3.1.

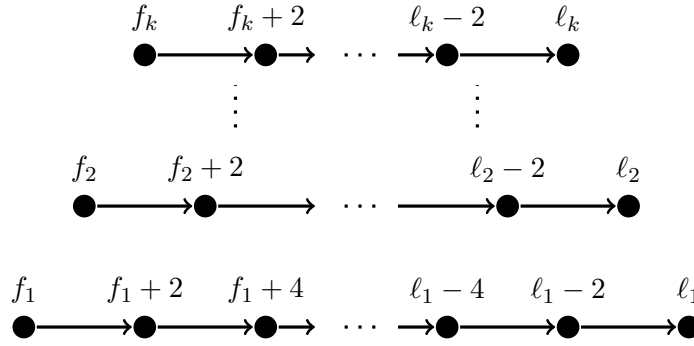


Figure 3.1: The nilpotent map e_P , where an arrow from $f_i + 2p$ to $f_j + 2q$ denotes that the standard basis vector corresponding to $f_i + 2p$ (i.e. the vector e_m , where m is the label of the box centred at $(f_i + 2p, 2i - 1)$) is mapped to the basis vector corresponding to $f_j + 2q$.

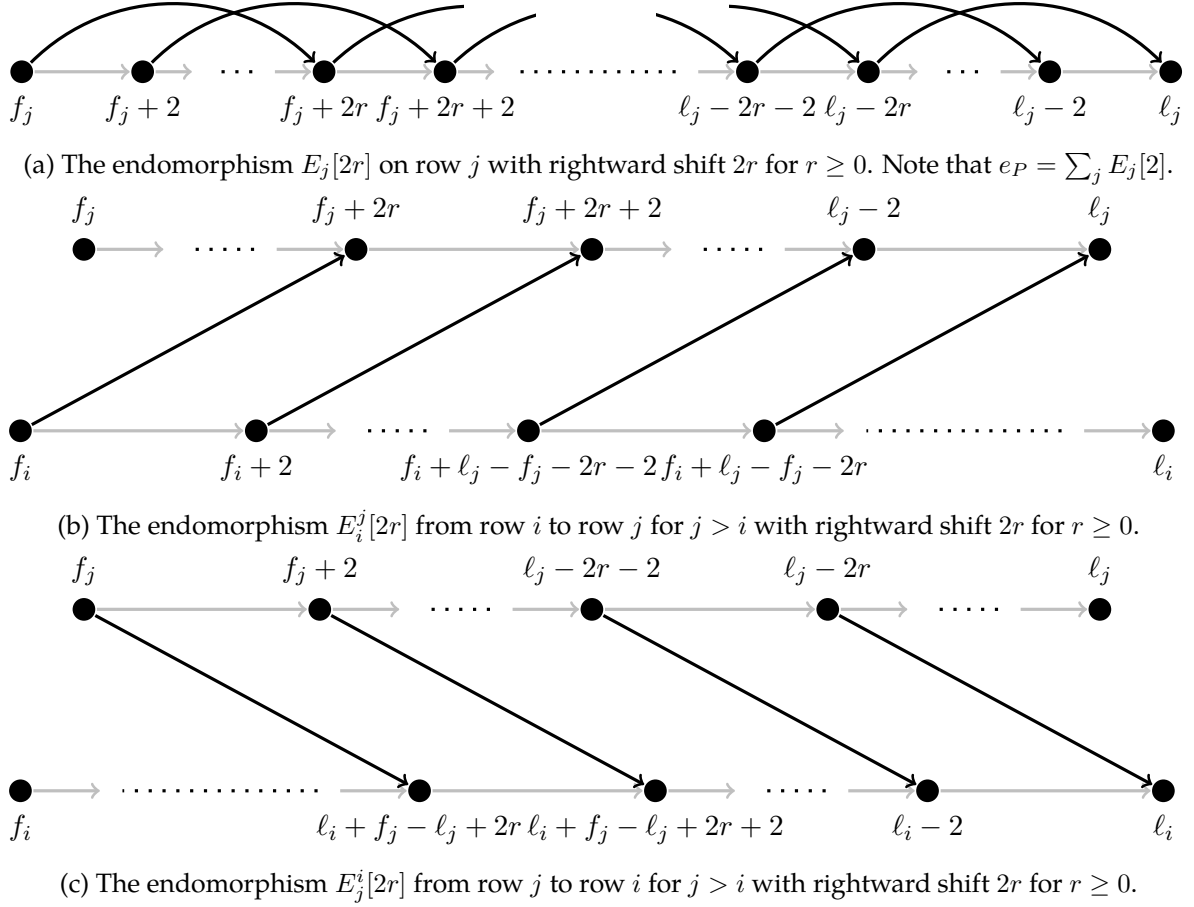
Figure 3.2 contains a collection of endomorphisms, represented by arrow diagrams, which commute with e_P . The endomorphisms in fig. 3.2a are positively graded by construction, and property 4 of definition 3.2.1, i.e. the pyramid condition of the definition, ensures that the endomorphisms in figs. 3.2b and 3.2c are also positively graded. The remainder of this proof consists of showing these endomorphisms form a basis of $\mathfrak{z}(e_P)$. They are linearly independent by construction, so it remains only to show they span the kernel.

Let $\lambda = (\lambda_1, \dots, \lambda_k)$ be the shape of P , and let $\lambda^* = (\lambda_1^*, \dots, \lambda_\ell^*)$ be the dual partition. Recall that the dimension of $\mathfrak{z}(e_P)$ is $\sum_{i=1}^\ell \lambda_i^{*2}$, as any endomorphism which commutes with e_P has a simultaneous Jordan basis, and hence is related to e_P by a collection of endomorphisms of the spaces generated by the i th elements of the Jordan strings.

Counting the endomorphisms of fig. 3.2, we can see that there are $\sum_{i=1}^k \lambda_i$ of those in fig. 3.2a, and $\sum_{i=1}^k \sum_{j=i+1}^k \lambda_j = \sum_{i=1}^k (i-1)\lambda_i$ each of those in figs. 3.2b and 3.2c. Hence the dimension of the space spanned by the endomorphisms is

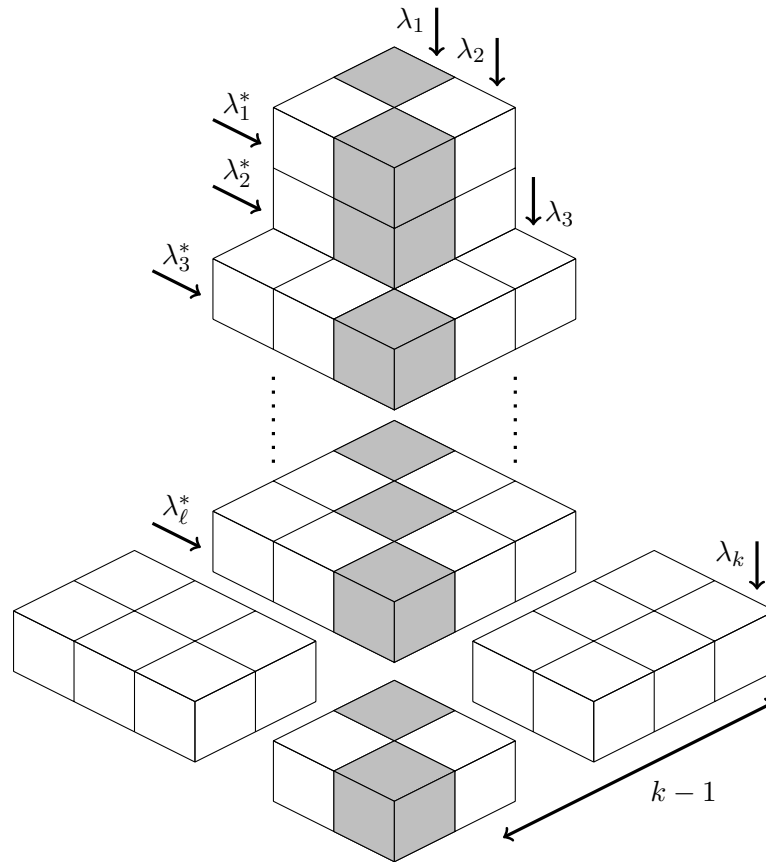
$$\sum_{i=1}^k \lambda_i + 2 \sum_{i=1}^k (i-1)\lambda_i = n + 2 \sum_{i=1}^k (i-1)\lambda_i = \sum_{i=1}^{\lambda_1} \lambda_i^{*2},$$

where the second equality follows by counting cubes in a tower of blocks associated to λ as in fig. 3.3. Thus the endomorphisms of fig. 3.2 span, and hence form a basis for, $\mathfrak{z}(e_P)$. It therefore follows that Γ_P is a good grading for e_P . \square



Note. The endomorphism e_P is also shown in each diagram, consisting of the lighter horizontal lines. It can be checked from the diagram that all such endomorphisms commute with e_P .

Figure 3.2: Endomorphisms of \mathbb{C}^n commuting with e_P .



$$n + 2 \sum_{i=1}^k (i-1) \lambda_i = \sum_{i=1}^{\ell} \lambda_i^{*2} \quad (3.2)$$

Figure 3.3: A tower of blocks associated to the partition $\lambda = (\lambda_1, \dots, \lambda_k)$ of n , with dual partition $\lambda^* = (\lambda_1^*, \dots, \lambda_{\ell}^*)$. This demonstrates the identity eq. (3.2): the number of blocks in the i th horizontal plane is λ_i^{*2} , the number of shaded blocks is n , and the number of blocks in the vertical planes on each side is $(i-1)\lambda_i$.

Example 3.2.5. For pyramid eq. (3.1), the corresponding nilpotent and good grading are

$$e_P = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad \Gamma_P: \begin{pmatrix} 0 & 1 & 1 & 2 & 3 & 3 & 4 \\ -1 & 0 & 0 & 1 & 2 & 2 & 3 \\ -1 & 0 & 0 & 1 & 2 & 2 & 3 \\ -2 & -1 & -1 & 0 & 1 & 1 & 2 \\ -3 & -2 & -2 & -1 & 0 & 0 & 1 \\ -3 & -2 & -2 & -1 & 0 & 0 & 1 \\ -4 & -3 & -3 & -2 & -1 & -1 & 0 \end{pmatrix},$$

where we here denote the grading Γ_P using a matrix whose (i, j) -entry is $\deg E_{ij}$.

It therefore follows that every pyramid corresponds, using some filling and choice of standard basis, to a pair of a nilpotent element and good grading (e, Γ) . Choosing a different filling or a different standard basis will produce a different pair (e', Γ') , however the two will be conjugate under some change of basis. Hence a pyramid corresponds to a conjugacy class of pairs (e, Γ) under the adjoint action of G . It remains to ask whether every conjugacy class of pairs (e, Γ) with Γ good for e comes from some pyramid. This question was also answered affirmatively by Elashvili and Kac.

Theorem 3.2.6 (Elashvili–Kac). [EK, Theorem 4.2] *There is a bijection between the pyramids of size n and the set of pairs (e, Γ) up to conjugacy, where $e \in \mathfrak{gl}_n$ is a nilpotent element and Γ is a good grading for e .*

$$\begin{aligned} \{\text{Pyramids of size } n\} &\leftrightarrow \{(e, \Gamma) : \Gamma \text{ is a good grading for } e\} / GL_n \\ P &\mapsto (e_P, \Gamma_P) \end{aligned}$$

This holds equally well for \mathfrak{sl}_n .

Remark 3.2.7. The pyramids which are symmetric under reflection about the zero column are called *symmetric pyramids*, and correspond to Dynkin gradings under this bijection. *Even pyramids*, those for which $\text{col}(j) - \text{col}(i)$ is even for every pair of boxes j and i , correspond to even good gradings. Since not every pyramid is symmetric, this demonstrates that not every good grading is Dynkin. Furthermore, since there always exists an even pyramid of given shape (for example, choosing the pyramid for which all rows are right-aligned), there always exists an even good grading for any nilpotent e .

3.3 Hamiltonian reduction by stages

In section 2.3, we discussed the relationship between Slodowy slices and Hamiltonian reduction. In particular it was shown that, for any good grading Γ , the nilpotent $e \in \mathfrak{g}$ can be completed to a Γ -graded \mathfrak{sl}_2 -triple $\{e, h, f\}$, and that the Slodowy slice $\mathcal{S}_e := e + \mathfrak{z}(f)$ can be expressed as a Hamiltonian reduction of the Poisson variety \mathfrak{g}^* under the co-adjoint action of the unipotent group $M \subseteq G$, after identifying \mathfrak{g} with \mathfrak{g}^* . However, \mathfrak{g}^* is itself a Slodowy slice \mathcal{S}_0 , taking the trivial grading Γ with $\mathfrak{g}_0 = \mathfrak{g}$ and the Γ -graded triple $h = e = f = 0$. Given two different Slodowy slices \mathcal{S}_e and $\mathcal{S}_{e'}$ with associated good gradings and \mathfrak{sl}_2 -triples, there is therefore a pair of reductions:

$$\begin{array}{ccc} \mathcal{S}_0 & & \\ \text{Reduction by } M \swarrow & \text{Reduction by } M' \searrow & \\ \mathcal{S}_e & & \mathcal{S}_{e'} \end{array}$$

We shall see that this diagram can actually be completed in type A under certain conditions on the nilpotents e and e' , with $\mathcal{S}_{e'}$ expressible as a Hamiltonian reduction of \mathcal{S}_e by the action of some unipotent group U :

$$\begin{array}{ccc}
 \mathcal{S}_0 & & \\
 \text{Reduction by } M \swarrow & \text{Reduction by } M' \searrow & \\
 \mathcal{S}_e & \xrightarrow{\text{Reduction by } U} & \mathcal{S}_{e'}
 \end{array} \tag{3.3}$$

This type of procedure, decomposing a Hamiltonian reduction into a sequence of smaller reductions, is known as *Hamiltonian reduction by stages*. This is a general technique which applies in the context of any Poisson variety, and in particular does not require us to assume that our Poisson variety is \mathfrak{g}^* , for \mathfrak{g} a simple Lie algebra of type A. A general reference for this material can be found in [MMO⁺].

3.3.1 Semidirect products and reduction by stages

Let G be an algebraic group which can be expressed as a semidirect product $G \simeq H \rtimes K$, where H and K are closed subgroups and H is normal in G . Let X be a Poisson variety on which G acts, and assume that the action is Hamiltonian with equivariant moment map $\mu: X \rightarrow \mathfrak{g}^*$. Under these circumstances, one can consider the Hamiltonian reduction of X by the action of G at a regular value $\gamma \in \mathfrak{g}^*$ of μ ; this shall be denoted $X //_{\gamma} G := \mu^{-1}(\gamma)/G$.

Since G decomposes as a semidirect product $H \rtimes K$, the closed H also acts on X by the inclusion of H into G . This action is also Hamiltonian, and its moment map $\mu_H: X \rightarrow \mathfrak{h}^*$ is the composition of the moment map μ with the restriction of functions $j: \mathfrak{g}^* \rightarrow \mathfrak{h}^*$. The subgroup K also has this property, but this is not important for our purposes. Note that μ_H is equivariant not only with respect to the action of H , but also with the respect to the action of G , as both μ and j are G -equivariant maps.

Identifying \mathfrak{g}^* with $\mathfrak{h}^* \times \mathfrak{k}^*$, the regular value $\gamma \in \mathfrak{g}^*$ can be decomposed as $\gamma = (\eta, \kappa)$, and further $\eta \in \mathfrak{h}^*$ is a regular value of the moment map μ_H . This allows one to consider the Hamiltonian reduction of X by H at the regular value η , $X //_{\eta} H := \mu_H^{-1}(\eta)/H$. We would like to relate $X //_{\gamma} G$ and $X //_{\eta} H$, and in particular would like to establish a Hamiltonian action of K on $X //_{\eta} H$ so that the subsequent reduction by this action produces a Poisson variety isomorphic to $X //_{\gamma} G$. This can be done under certain conditions on the group K and the values η and κ .

By the definition of the semidirect product, there is an action of K on H , and hence an induced action of K on \mathfrak{h}^* . We shall assume that the action of K stabilises $\eta \in \mathfrak{h}^*$. In this case, there is an induced action of K on $X //_{\eta} H$, where the well-definedness of the action follows from the normality of H , the G -equivariance of μ_H , and the fact that K stabilises η . Furthermore, this action is Hamiltonian with induced moment map $\mu_K: X //_{\eta} H \rightarrow \mathfrak{k}^*$ defined by $\mu_K([x]) = \mu(x)$. We finally assume that $\kappa \in \mathfrak{k}^*$ is a regular value of μ_K .

Theorem 3.3.1. *Let X be a Poisson variety with a Hamiltonian action by the algebraic group $G \simeq H \rtimes K$ satisfying the above hypotheses. There is a Poisson isomorphism between the space $X //_{\gamma} G$ and the two-stage reduction $(X //_{\eta} H) //_{\kappa} K$.*

Note. There are many versions of this theorem with varying sets of hypotheses. In particular, it holds in much greater generality, as in [MMO⁺, Theorem 5.2.9]. Though presented there for symplectic varieties, the proof for Poisson varieties follows identically making the necessary changes. Since we're making the simplifying assumptions that G is a semidirect product

$H \rtimes K$ for which K stabilises η , most of the details of the theorem simplify; in particular the *stages hypothesis* is automatically satisfied and the moment map of the action of K on $X //_{\eta} H$ is obtained by lifting to X and applying μ .

3.4 Reduction by stages for Slodowy slices

As has been previously established in theorem 2.3.4, the Slodowy slices \mathcal{S}_{χ} can be expressed as Hamiltonian reductions of the dual Lie algebra \mathfrak{g}^* . This can and will be equivalently stated in the Lie algebra itself, rather than its dual, by applying the Killing isomorphism: \mathcal{S}_e can be expressed as the Hamiltonian reduction of \mathfrak{g} . Since \mathfrak{g} is the Slodowy slice through the zero nilpotent, one might ask whether different Slodowy slices can be expressed as Hamiltonian reductions of other Slodowy slices. The objective of this section is to provide a conjecture regarding under which conditions this can be done, and to provide a construction to accomplish this.

Objective 3.4.1. *Let \mathfrak{g} be a Lie algebra of type A, and e_1 and e_2 be two nilpotent elements of \mathfrak{g} such that $\mathcal{O}_{e_1} < \mathcal{O}_{e_2}$, with Slodowy slices \mathcal{S}_{e_1} and \mathcal{S}_{e_2} , respectively. Then we would like to exhibit an algebraic group K with a Hamiltonian action on \mathcal{S}_{e_1} , along with a regular value κ of the moment map $\mu: \mathcal{S}_{e_1} \rightarrow \mathfrak{k}^*$, such that \mathcal{S}_{e_2} can be expressed as a Hamiltonian reduction of \mathcal{S}_{e_1} , i.e. $\mathcal{S}_{e_2} \simeq \mathcal{S}_{e_1} //_{\kappa} K$.*

This will produce a collection of commuting reductions of Slodowy slices for every edge in the Hasse diagram of the partial ordering on nilpotent orbits, as in fig. 1.3. Since every pair of nilpotent orbits $\mathcal{O}_1 < \mathcal{O}_2$ can be filled in by a sequence of covering relations $\mathcal{O}_1 < \dots < \mathcal{O}_2$, it will suffice to construct the reductions assuming that \mathcal{O}_{e_1} is covered by \mathcal{O}_{e_2} . Theorem 3.3.1 provides the tools needed to construct this. For any pair of nilpotent orbits with \mathcal{O}_1 covering \mathcal{O}_2 , we will choose

- nilpotent elements $e_1 \in \mathcal{O}_1$ and $e_2 \in \mathcal{O}_2$ with Killing duals χ_1 and χ_2 , respectively;
- a good grading Γ_1 for e_1 with Premet subalgebra \mathfrak{m}_1 and algebraic group M_1 ; and
- a subalgebra $\mathfrak{m}_2 \supseteq \mathfrak{m}_1$ with corresponding algebraic group M_2

which satisfy the following conditions:

- SR1. the subalgebra \mathfrak{m}_2 decomposes as a semidirect product $\mathfrak{m}_2 = \mathfrak{m}_1 \rtimes \mathfrak{k}$;
- SR2. the functional χ_2 restricts to a character of \mathfrak{m}_2 and decomposes as (χ_1, κ) in the above decomposition;
- SR3. the subalgebra \mathfrak{k} annihilates χ_1 ; and
- SR4. the value $\kappa \in \mathfrak{k}^*$ is a regular value of the moment map $\mu_K: \mathfrak{g}^* //_{\chi_1} M_1 \rightarrow \mathfrak{k}^*$ of the action of K , the algebraic group corresponding to \mathfrak{k} , on the Hamiltonian reduction $\mathfrak{g}^* //_{\chi_1} M_1$, and the action of K on $\mu_K^{-1}(\kappa)$ is free and proper.

With these choices it follows that M_2 decomposes as a semidirect product $M_2 = M_1 \rtimes K$, and there is a corresponding reduction by stages construction

$$X = \mathfrak{g}^* //_{\chi_2} M_2 \simeq (\mathfrak{g}^* //_{\chi_1} M_1) //_{\kappa} K \simeq \mathcal{S}_{\chi_1} //_{\kappa} K$$

provided by theorems 2.3.4 and 3.3.1. We will therefore provide a construction satisfying these conditions, and conjecture that the Poisson variety X obtained is isomorphic to the Slodowy slice \mathcal{S}_{χ_2} . Provided this conjecture holds, this will accomplish objective 3.4.1.

Proposition 3.4.2. *If \mathfrak{m}_2 is a Premet subalgebra for a good grading of e_2 then $X \simeq \mathcal{S}_{\chi_2}$. Furthermore, conditions SR1 and SR2 imply conditions SR3 and SR4. Objective 3.4.1 therefore follows under these conditions.*

Proof. Since Slodowy slices can be expressed as Hamiltonian reductions of \mathfrak{g}^* by Premet subgroups, that $\mathcal{S}_{\chi_2} \simeq \mathfrak{g}^* //_{\chi_2} M_2$ follows directly from theorem 2.3.4.

Recall that by proposition 2.2.3, χ_1 and χ_2 are characters of \mathfrak{m}_1 and \mathfrak{m}_2 , respectively. Therefore χ_2 vanishes on $[\mathfrak{m}_2, \mathfrak{m}_2]$, and so is annihilated by \mathfrak{m}_2 in general, and $\mathfrak{k} \subseteq \mathfrak{m}_2$ in particular, establishing ??cond:KStabilise.

To prove ??cond:KRegular, note that $\mu_K: \mathcal{S}_{\chi_1} \rightarrow \mathfrak{k}^*$ has tangent map

$$T_{[\xi]} \mu_K: T_{[\xi]}(\mathfrak{g}^* //_{\chi_1} M_1) \longrightarrow T_{\xi|_{\mathfrak{k}}} \mathfrak{k}^*.$$

This map, expressed more concretely, is the restriction of functions $\text{res}: \mathfrak{m}_1^{*,\perp} / \mathfrak{m}_1^* \rightarrow \mathfrak{k}^*$. This is well-defined as $\mathfrak{k} \subseteq \mathfrak{m}_2$, and therefore $\mathfrak{k}^* \subseteq \mathfrak{m}_2^{*,\perp} \subseteq \mathfrak{m}_1^{*,\perp}$, and surjective as $\mathfrak{k} \cap \mathfrak{m}_1 = \{0\}$. That the action is locally free and proper follows from the freeness and properness of the action of M_2 . \square

Note. Keeping in mind the parallels between Slodowy slices and W-algebras, these constructions should be applicable not only to Hamiltonian reduction of Slodowy slices, but also to quantum Hamiltonian reduction of W-algebras. We shall therefore work with Lie algebras instead of algebraic groups, and the corresponding statements for groups follow directly by exponentiation.

3.4.1 A construction in examples

Before providing the general construction satisfying conditions SR1 to SR4, it will be useful to see the construction in a number of examples. The main features can be seen in some concrete cases, and the general construction follows in a straightforward manner from these examples.

Reducing from a subregular nilpotent in \mathfrak{sl}_3

Let $\mathfrak{g} = \mathfrak{sl}_3$, and let \mathcal{O}_1 and \mathcal{O}_2 be the subregular and regular nilpotent orbits, respectively. First, we construct a right-aligned pyramid for the subregular element in Jordan canonical form. Note that a right-aligned pyramid is automatically even, and so uniquely specifies a Premet subalgebra.

$$P_1 = \begin{array}{|c|c|} \hline & 3 \\ \hline 1 & 2 \\ \hline \end{array} \quad e_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \mathfrak{m}_1 = \begin{pmatrix} 0 & 0 & 0 \\ * & 0 & 0 \\ * & 0 & 0 \end{pmatrix}$$

If box 3 were moved to the bottom row, the resulting pyramid would be the standard pyramid for the regular nilpotent element in \mathfrak{g} in Jordan canonical form, e_2 .

$$P_2 = \begin{array}{|c|c|c|} \hline 1 & 2 & 3 \\ \hline \end{array} \quad e_2 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \quad \mathfrak{m}_2 = \begin{pmatrix} 0 & 0 & 0 \\ * & 0 & 0 \\ * & * & 0 \end{pmatrix}$$

Since \mathfrak{m}_2 is a Premet subalgebra, it suffices to check conditions SR1 and SR2. One can check that \mathfrak{m}_1 is an ideal of \mathfrak{m}_2 , and the complementary subalgebra \mathfrak{k} can be chosen to be

$$\mathfrak{k} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & * & 0 \end{pmatrix},$$

verifying condition SR1. Condition SR2 can be checked by observing that $\chi_2|_{\mathfrak{m}_1} = \chi_1$.

Reducing from the middle nilpotent in \mathfrak{sl}_4

Let $\mathfrak{g} = \mathfrak{sl}_4$ and let \mathcal{O}_1 be the middle nilpotent orbit, i.e. the orbit consisting of all nilpotent elements of type (2,2). This covers the subregular nilpotent orbit \mathcal{O}_2 . We again construct a right-aligned pyramid for a middle nilpotent, specifying a unique Premet subalgebra.

$$P_1 = \begin{array}{|c|c|} \hline 2 & 4 \\ \hline 1 & 3 \\ \hline \end{array} \quad e_1 = E_{13} + E_{24} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \mathfrak{m}_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ * & * & 0 & 0 \\ * & * & 0 & 0 \end{pmatrix}$$

By sliding box 4 to the bottom row we would obtain a subregular nilpotent of type (3,1), however instead of choosing this pyramid we shall construct a subregular nilpotent and good grading as follows:

- Let the subregular nilpotent element e_2 be the sum of the original nilpotent e_1 and all matrices E_{ij} where i is a box in the first row and j is the box immediately above it in the second row, i.e. $\text{row}(i) = 1$, $\text{row}(j) = 2$ and $\text{col}(i) = \text{col}(j)$.

$$e_2 = e_1 + E_{12} + E_{34} = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

- Let \mathfrak{m}_2 be the Lie algebra generated by \mathfrak{m}_1 with the additional generator $E_{21} + E_{43}$, that is the sum of all E_{ij} such that $\text{row}(i) = 2$, $\text{row}(j) = 1$ and $\text{col}(i) = \text{col}(j)$.

$$\mathfrak{m}_2 = \left\{ \begin{pmatrix} 0 & 0 & 0 & 0 \\ a & 0 & 0 & 0 \\ * & * & 0 & 0 \\ * & * & a & 0 \end{pmatrix} : a \in \mathbb{C} \right\}.$$

A direct computation shows that conditions SR1 to SR4 are satisfied, with the choice of complementary subalgebra

$$\mathfrak{k} = \langle E_{21} + E_{43} \rangle = \left\{ \begin{pmatrix} 0 & 0 & 0 & 0 \\ a & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & a & 0 \end{pmatrix} : a \in \mathbb{C} \right\}.$$

Reducing from the zero nilpotent in \mathfrak{sl}_3

Let $\mathfrak{g} = \mathfrak{sl}_3$ and let $\mathcal{O}_1 = \{0\}$ be the zero orbit and \mathcal{O}_2 be the minimal orbit (which is also the subregular orbit).

$$P_1 = \begin{array}{|c|} \hline 3 \\ \hline 2 \\ \hline 1 \\ \hline \end{array} \quad e_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \mathfrak{m}_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

If box 3 were moved to the bottom row, the result would be a pyramid corresponding to a minimal nilpotent element. The nilpotent e_2 and subalgebra \mathfrak{m}_2 shall be chosen as follows:

- Let e_2 be the sum of the original nilpotent e_1 and all matrices E_{ij} where i is in the first row and j is the box immediately above it in the third row, i.e. $\text{row}(i) = 1$, $\text{row}(j) = 3$ and $\text{col}(i) = \text{col}(j)$.

$$e_2 = e_1 + E_{13} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

- Let \mathfrak{m}_2 be the Lie algebra generated by \mathfrak{m}_1 with the additional generators $E_{21} + E_{32}$ and E_{31} . These are the generators corresponding to the integers $k = 1, 2$, where the generator corresponding to k consists of the sum of all E_{ij} such that $\text{row}(i) \leq 3$, $\text{row}(j) \geq 1$, $\text{col}(i) = \text{col}(j)$ and $\text{row}(i) - \text{row}(j) = k$.

$$\mathfrak{m}_2 = \langle E_{21} + E_{32}, E_{31} \rangle = \left\{ \begin{pmatrix} 0 & 0 & 0 \\ a & 0 & 0 \\ b & a & 0 \end{pmatrix} : a, b \in \mathbb{C} \right\}.$$

3.4.2 The general construction

Let $\mathfrak{g} = \mathfrak{sl}_n$ be the Lie algebra of type A_{n-1} . The conjugacy classes of nilpotent elements correspond to Jordan types, and are hence indexed by partitions of n . Consider a pair of nilpotent conjugacy classes indexed by $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_m)$ and $\mu = (\mu_1, \dots, \mu_{m+1})$ where λ covers μ , i.e. $\lambda > \mu$ and no partition lies intermediate of the two. Let i and j be the integers used for obtaining μ from λ as in proposition 3.1.3.

Construct a right-aligned pyramid for μ in the usual way, numbering the boxes from bottom to top and left to right. This determines a nilpotent $e_1 \in \mathcal{O}_\mu$ and $\mathfrak{m}_1 \subseteq \mathfrak{g}$, and hence determines a Slodowy slice \mathcal{S}_e . What remains is to choose $e_2 \in \mathcal{O}_\lambda$ and \mathfrak{m}_2 which satisfy conditions SR1 to SR4.

Theorem 3.4.3. *In the above circumstances, let e_2 and $\mathfrak{m}_2 \subseteq \mathfrak{g}$ be as follows:*

$$e_2 = e_1 + \sum_{\substack{\text{row}(k)=i, \text{row}(\ell)=j \\ \text{col}(k)=\text{col}(\ell)}} E_{k\ell} \quad \text{and} \quad \mathfrak{m}_2 = \mathfrak{m}_1 + \langle E_m \rangle_{m=1}^{j-i}, \quad \text{where} \quad E_m = \sum_{\substack{i \leq \text{row}(k) < \text{row}(\ell) \leq j \\ \text{row}(\ell) - \text{row}(k) = m \\ \text{col}(k) = \text{col}(\ell)}} E_{\ell k}.$$

Further, let $\chi_k = \langle e_k, \cdot \rangle$ for $k = 1, 2$ and $\mathfrak{k} = \langle E_m \rangle_{m=1}^{j-i}$. Then e_2 is a nilpotent element of Jordan type λ in \mathcal{O}_λ , and conditions SR1 to SR4 hold.

Remark 3.4.4. Note that the nilpotent e_2 and the generators E_m can be rewritten using the generators of $\mathfrak{z}(e_1)$ as in fig. 3.2c:

$$e_2 = e_1 + E_i^j[0] \quad E_m = \sum_{i \leq k \leq j-m} E_{k+m}^k[0]. \quad (3.4)$$

In particular, it can be seen from the figure that the generators E_m commute amongst themselves, and hence \mathfrak{k} is an abelian Lie algebra. To see that \mathfrak{m}_2 is indeed closed under the Lie bracket, observe that, using the good grading from the pyramid for μ ,

$$[\mathfrak{m}_2, \mathfrak{m}_2] = [\mathfrak{m}_2, \mathfrak{m}_1] \subseteq \left[\bigoplus_{k \leq 0} \mathfrak{g}_k, \bigoplus_{\ell \leq -2} \mathfrak{g}_\ell \right] \subseteq \bigoplus_{k \leq -2} \mathfrak{g}_k = \mathfrak{m}_1 \subseteq \mathfrak{m}_2.$$

Proof. To prove that e_2 has the correct Jordan type, it suffices to exhibit a Jordan basis. Note that a Jordan basis can be read off the rows of the pyramid, proceeding from left to right. Let the Jordan basis for e_1 in row i be given by

$$e_{i1} \mapsto e_{i2} \mapsto \cdots \mapsto e_{i\mu_i} \mapsto 0.$$

The Jordan basis for e_2 is identical to that of e_1 except for those strings corresponding to rows i and j . Specifically, we have the two strings

$$\begin{aligned} (e_{i1}) \mapsto \cdots \mapsto (e_{i\mu_i - \mu_j}) \mapsto (e_{i\mu_i - \mu_j + 1} + e_{j1}) \mapsto \cdots \mapsto (e_{i\mu_i} + (\mu_j - 1)e_{j\mu_j - 1}) \mapsto (\mu_j e_{j\mu_j}) \mapsto 0, \\ (e_{i\mu_i - \mu_j + 1} - (\mu_j - 1)e_{j1}) \mapsto (e_{i\mu_i - \mu_j + 2} - (\mu_j - 2)e_{j2}) \mapsto \cdots \mapsto (e_{i\mu_i} - e_{j\mu_j - 1}) \mapsto 0, \end{aligned}$$

of lengths $\mu_i + 1 = \lambda_i$ and $\mu_j - 1 = \lambda_j$, respectively.

Example 3.4.5. Consider partitions $\mu = (3, 2)$ and $\lambda = (4, 1)$. Choosing a right-aligned pyramid for μ , one obtains the pyramid and Jordan basis of type μ

$$\begin{array}{|c|c|} \hline & 3 \quad 5 \\ \hline 1 & 2 \quad 4 \\ \hline \end{array} \quad \begin{aligned} e_1 \mapsto e_2 \mapsto e_4 \mapsto 0 \\ e_3 \mapsto e_5 \mapsto 0. \end{aligned}$$

The corresponding Jordan basis of type λ is

$$\begin{aligned} (e_1) \mapsto (e_2) \mapsto (e_4 + e_3) \mapsto (2e_5) \mapsto 0 \\ (e_4 - e_3) \mapsto 0. \end{aligned}$$

Condition **SR1**, that $\mathfrak{m}_2 \simeq \mathfrak{m}_1 \rtimes \mathfrak{k}$, follows by showing that \mathfrak{m}_1 is a Lie ideal of \mathfrak{m}_2 and \mathfrak{k} is a complementary subalgebra, both of which are shown in remark 3.4.4.

To check condition **SR2**, it can be seen that $\chi_2|_{\mathfrak{m}_1} = \chi_1$ by construction. To confirm that χ_2 is a character of \mathfrak{m}_2 , note that $[\mathfrak{m}_2, \mathfrak{m}_2] = [\mathfrak{m}_1 + \mathfrak{k}, \mathfrak{m}_1]$ by remark 3.4.4. However, since $\chi_2([\mathfrak{m}_1, \mathfrak{m}_1]) = \chi_1([\mathfrak{m}_1, \mathfrak{m}_1]) = 0$, it remains only to check that $\chi_2([\mathfrak{k}, \mathfrak{m}_1]) = 0$. We shall check this on a basis $\{[E_m, E_{\ell k}] : \text{col}(k) < \text{col}(\ell), 1 \leq m \leq j - i\}$. First, note that

$$\chi_2([E_m, E_{\ell k}]) = \langle e_2, [E_m, E_{\ell k}] \rangle = \langle [e_2, E_m], E_{\ell k} \rangle.$$

This vanishes, as eq. (3.4) implies that

$$[e_2, E_m] = \left[e_1 + E_i^j[0], \sum_{i \leq k \leq j-m} E_{k+m}^k[0] \right] = 0.$$

This further establishes the stronger claim that \mathfrak{k} annihilates χ_2 , and also χ_1 : hence condition **SR3** also holds. Condition **SR4** follows by the same argument as in proposition 3.4.2. \square

Conjecture 3.4.6. For nilpotents $e_1, e_2 \in \mathfrak{g}$ and subalgebras $\mathfrak{m}_1, \mathfrak{m}_2 \subseteq \mathfrak{g}$ as defined in theorem 3.4.3, the reduced space $\mathfrak{g}^* //_{\chi_2} M_2$ is isomorphic to the Slodowy slice \mathcal{S}_{χ_2} as a Poisson variety.

Remark 3.4.7. This conjecture is a special case of a more general conjecture due to Premet, based on his work in [Pre1] (cf. [Sad, Question 1]). Specifically, Premet conjectures that for any ad-nilpotent subalgebra $\mathfrak{m} \subseteq \mathfrak{g}$ satisfying the conditions:

1. $\chi(\mathfrak{m}, \mathfrak{m}) = 0$, i.e. χ is a character of \mathfrak{m} ;
2. $\mathfrak{m} \cap \mathfrak{z}_{\mathfrak{g}}(e) = \{0\}$; and
3. $\dim \mathfrak{m} = (\dim G \cdot e)/2$;

the W-algebras $U(\mathfrak{g}, e)$, and therefore the classical reduced spaces $\mathfrak{g} //_{\chi} M$, are all isomorphic.

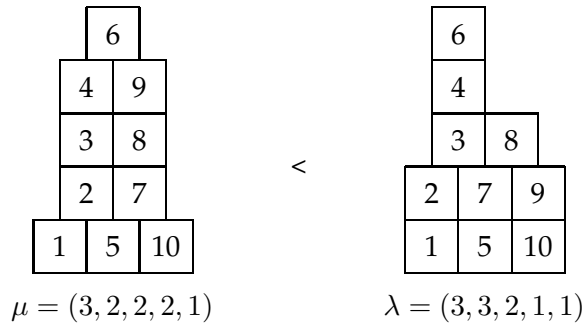
Concretely, the algebra \mathfrak{m}_2 defined in theorem 3.4.3 satisfies conditions 1 and 3 by construction, and it can be checked that it satisfies condition 2 by modifying the diagrams in fig. 3.2. Premet's conjecture would therefore imply that the Hamiltonian reduction by \mathfrak{m}_2 at χ_2 is isomorphic to the Slodowy slice \mathcal{S}_{χ_2} .

In fact, Premet has proven this conjecture in the case that the base field is of characteristic p [Pre1]. Hence conjecture 3.4.6, and the construction presented here in general, is actually a theorem when working over a field of non-zero characteristic.

Proposition 3.4.8. Conjecture 3.4.6 holds for e_1 a subregular nilpotent and e_2 a regular nilpotent.

Proof. The subalgebra \mathfrak{m}_2 constructed is simply the maximal nilpotent subalgebra of lower-triangular matrices \mathfrak{n}_- . This is a Premet subalgebra for e_2 . \square

Remark 3.4.9. The construction detailed in this section can be modified slightly to give a stronger version of proposition 3.4.8. Instead of choosing a right-aligned pyramid of shape μ , one can choose a pyramid which is right-aligned but for a leftward shift of 1 row i and another leftward shift of 1 at row $j + 1$. This necessitates a choice of Lagrangian $\mathfrak{l} \subseteq \mathfrak{g}_{-1}$; this choice can be made so that the resulting Premet subalgebra can be extended to a Premet subalgebra for a pyramid of shape λ which is right-aligned but for a leftward shift of 1 at row $i + 1$ and another leftward shift of 2 at row j .



For this new pyramid and compatible choice of Lagrangian, theorem 3.4.3 remains true. Furthermore, proposition 3.4.8 and its proof hold not only for e_1 a subregular nilpotent and e_2 a regular nilpotent, but more generally for any pair of nilpotent elements e_1 and e_2 of Jordan types $\mu = (\mu_1, \dots, \mu_k, 1)$ and $\lambda = (\mu_1, \dots, \mu_k + 1)$, respectively.

Example 3.4.10. Let $\mathfrak{g} = \mathfrak{sl}_4$, \mathcal{O}_1 be the middle nilpotent orbit and \mathcal{O}_2 be the subregular nilpotent orbit as in page 35. The Slodowy slice \mathcal{S}_{χ_2} and a presentation for the reduced space $\mathfrak{g}^* //_{\chi_2} M_2$

thus obtained are:

$$\mathcal{S}_{\chi_2} = \left\{ \begin{pmatrix} a & 1 & 0 & 0 \\ b - 3a^2 & a & 1 & 0 \\ c + 20a^3 & b - 3a^2 & a & d \\ f & 0 & 0 & -3a \end{pmatrix} : a, b, c, d, f \in \mathbb{C} \right\}, \quad \mathbb{C}[\mathcal{S}_{\chi_2}] = \mathbb{C}[a, b, c, d, f]$$

$$\mathfrak{g}^*_{\chi_2} M_2 \simeq \left\{ \begin{pmatrix} 0 & 1 & 1 & 0 \\ x + \frac{u+v}{4} & 0 & 0 & 1 \\ \frac{-3u+v}{4} & -2y & 0 & 1 \\ z + \frac{u+v}{2}y & \frac{u-3v}{4} & x + \frac{u+v}{4} & 0 \end{pmatrix} : u, v, x, y, z \in \mathbb{C} \right\}, \quad \mathbb{C}[\mathfrak{g}^*_{\chi_2} M_2] = \mathbb{C}[u, v, x, y, z].$$

The non-zero Poisson brackets are given by the formulae:

$$\begin{aligned} \{a, d\} &= \frac{-1}{24}d & \{c, d\} &= \frac{1}{6}bd & \{u, y\} &= \frac{1}{8}(u + x + y^2) & \{u, z\} &= \frac{1}{4}x(u + x + y^2) \\ \{a, f\} &= \frac{1}{24}f & \{c, f\} &= \frac{-1}{6}bf & \{v, y\} &= \frac{-1}{8}(v + x + y^2) & \{v, z\} &= \frac{-1}{4}x(v + x + y^2) \\ \{d, f\} &= \frac{-27}{2}a^3 + ab - \frac{1}{8}c & \{u, v\} &= \frac{-1}{4}(z + xy + 2(u + v)y) \end{aligned}$$

Consider the ring homomorphism $\varphi: \mathbb{C}[\mathcal{S}_{\chi_2}] \rightarrow \mathbb{C}[\mathfrak{g}^*_{\chi_2} M_2]$ defined on generators by

$$\varphi(a) = \frac{-1}{3}y \quad \varphi(b) = x \quad \varphi(c) = 2z - \frac{8}{3}xy \quad \varphi(d) = v + x + y^2 \quad \varphi(f) = -u - x - y^2.$$

It can be checked that this map is a ring isomorphism and also preserves the Poisson bracket; it hence induces an isomorphism of the Poisson varieties $\mathcal{S}_{\chi_2} \simeq \mathfrak{g}^*_{\chi_2} M_2$. Furthermore, this map preserves the characteristic polynomial, and is the unique map satisfying all these properties, up to the automorphism α of $\mathbb{C}[\mathcal{S}_{\chi_2}]$ ($\alpha(d) = -d$, $\alpha(f) = -f$, and all other generators fixed).

3.5 Quantum Hamiltonian reduction by stages for W-algebras

The construction of section 3.4.2 gives a method for constructing semidirect product decompositions $M_2 \simeq M_1 \rtimes K$ suitable for constructing Slodowy slices as Hamiltonian reductions of Slodowy slices corresponding to more singular nilpotents. However, it has been presented in such a way that its generalisation to the quantum case and W-algebras is straightforward. In particular, we shall construct a notion of *quantum Hamiltonian reduction by stages* which generalises the classical Hamiltonian reduction by stages, and present a construction expressing W-algebras as quantum Hamiltonian reductions of W-algebras corresponding to more singular nilpotent elements. What follows is a quantum version of theorem 3.3.1, the Hamiltonian reduction by stages theorem.

Theorem 3.5.1. *Let \mathfrak{g} be a Lie algebra with universal enveloping algebra $U(\mathfrak{g})$, and let \mathfrak{m}_1 , \mathfrak{m}_2 and \mathfrak{k} be ad-nilpotent subalgebras of \mathfrak{g} . Furthermore, let $\mathfrak{m}_2 = \mathfrak{m}_1 \rtimes \mathfrak{k}$, and let χ_2 be a character of \mathfrak{m}_2 which decomposes as $\chi_2 = (\chi_1, \kappa)$. Denoting by \mathfrak{m}_{1, χ_1} , \mathfrak{m}_{2, χ_2} and \mathfrak{k}_κ the corresponding shifted Lie algebras, define the quantum Hamiltonian reductions by*

$$U_1 := U(\mathfrak{g})_{\chi_1} \mathfrak{m}_1 = (U(\mathfrak{g})/U(\mathfrak{g})\mathfrak{m}_{1, \chi_1})^{\mathfrak{m}_1} \quad \text{and} \quad U_2 := U(\mathfrak{g})_{\chi_2} \mathfrak{m}_2 = (U(\mathfrak{g})/U(\mathfrak{g})\mathfrak{m}_{2, \chi_2})^{\mathfrak{m}_2},$$

where the invariants are equivalently either taken with respect to the adjoint action or left multiplication by the shifted Lie algebras. Then U_2 can be expressed as a quantum Hamiltonian reduction of U_1 :

$$U_2 \simeq (U_1/U_1\mathfrak{k}_\kappa)^{\mathfrak{k}}.$$

With this theorem, it is now possible to approach the problem of expressing W-algebras as intermediate quantum Hamiltonian reductions. Specifically, since the construction of section 3.4.2 is phrased in terms of Lie algebras and characters, one can apply this theorem directly to the construction to obtain a quantum Hamiltonian reduction of the W-algebra $U(\mathfrak{g}, e_1)$. Choosing \mathfrak{m}_1 to be a Premet subalgebra for e_1 , the algebra U_1 is just the W-algebra $U(\mathfrak{g}, e_1)$, and the construction gives a nilpotent e_2 with a subalgebra \mathfrak{m}_2 satisfying

$$U(\mathfrak{g}, e_1) \ll_{\kappa} \mathfrak{k} \simeq U(\mathfrak{g}) \ll_{\chi_2} \mathfrak{m}_2.$$

Conjecture 3.5.2. *The reduced space $U(\mathfrak{g}) \ll_{\chi_2} \mathfrak{m}_2$ is isomorphic to the W-algebra $U(\mathfrak{g}, e_2)$.*

This conjecture is a quantum version of conjecture 3.4.6, and its veracity is closely related. Specifically, since the W-algebras $U(\mathfrak{g}, e_i)$ are filtered algebras whose associated graded algebras are $\mathbb{C}[\mathcal{S}_{\chi_i}]$, to prove that $U(\mathfrak{g}, e_2) \simeq U(\mathfrak{g}) \ll_{\chi_2} \mathfrak{m}_2$ would require lifting an isomorphism $\varphi: \mathfrak{g} \ll_{\chi_2} M_2 \rightarrow \mathcal{S}_{\chi_2}$ to a homomorphism $\tilde{\varphi}: U(\mathfrak{g}, e_2) \rightarrow U(\mathfrak{g}) \ll_{\chi_2} \mathfrak{m}_2$, which would then automatically be an isomorphism by the general properties of filtered algebras.

Chapter 4

Category \mathcal{O} for W-algebras

The BGG category \mathcal{O} , hereafter referred to just as *category \mathcal{O}* , is a certain full subcategory of the category of all \mathfrak{g} -modules which satisfies three conditions. Given a triangular decomposition $\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$, category \mathcal{O} consists of all modules in \mathfrak{g} -mod which:

1. are finitely-generated,
2. are acted upon semi-simply by \mathfrak{h} ,
3. are acted upon locally nilpotently by \mathfrak{n}_+ .

This category was originally defined by Bernstein, Gel'fand and Gel'fand in [BGG], and has since proven extremely useful to mathematicians, in particular for its relation to a number of categorification constructions (cf. [BFK, KMS1]).

A similar subcategory exists in the category of all modules over a W-algebra $U(\mathfrak{g}, e)$. As $U(\mathfrak{g}, e)$ is a subquotient of $U(\mathfrak{g})$, it can be equipped with a similar triangular decomposition, and analogues of conditions 1 to 3 can be formulated. In [Los1], Losev investigates the structure of this analogue of category \mathcal{O} for W-algebras, and constructs an equivalence between it and a certain subcategory of \mathfrak{g} -mod.

The objective of this chapter is to prove a similar equivalence in type A, relating the categories \mathcal{O} for W-algebras to the categories \mathcal{O} for W-algebras for more singular nilpotent elements, using the quantum Hamiltonian reduction by stages construction developed in section 3.5. In particular, this chapter shall assume that conjecture 3.5.2 holds.

4.1 The Skryabin equivalence

To work with category \mathcal{O} for a W-algebra, it is first necessary to consider the category of all modules over a W-algebra: $U(\mathfrak{g}, e)$ -mod. Although W-algebras are difficult to work with on their own, the situation can be significantly simplified by constructing an equivalence between $U(\mathfrak{g}, e)$ -mod and a certain subcategory of \mathfrak{g} -mod corresponding to what are known as *Whittaker modules*.

Recall that the W-algebra $U(\mathfrak{g}, e)$ can be defined as the invariants under the adjoint action of \mathfrak{m} in the *generalised Gel'fand–Graev module* $Q_\chi := U(\mathfrak{g})/U(\mathfrak{g})\mathfrak{m}_\chi$.

$$U(\mathfrak{g}, e) = (Q_\chi)^\mathfrak{m}.$$

The definition of Q_χ allows one to note that it both has the usual left action of $U(\mathfrak{g})$ by multiplication, but it further has a well-defined right action of $U(\mathfrak{g}, e)$. Hence, Q_χ can be seen to

be a $(U(\mathfrak{g}), U(\mathfrak{g}, e))$ -bimodule, and therefore induces an adjoint pair of functors between the associated module categories:

$$\begin{aligned} Q_\chi \otimes_{U(\mathfrak{g}, e)} -: U(\mathfrak{g}, e)\text{-mod} &\rightarrow U(\mathfrak{g})\text{-mod} \\ \text{Hom}_{U(\mathfrak{g})}(Q_\chi, -) : U(\mathfrak{g})\text{-mod} &\rightarrow U(\mathfrak{g}, e)\text{-mod} \end{aligned}$$

Definition 4.1.1. The functor $Q_\chi \otimes_{U(\mathfrak{g}, e)} -$ is called the *Skryabin functor*.

The essential image of the Skryabin functor can be described in intrinsic terms: it is the full subcategory of $U(\mathfrak{g})\text{-mod}$ consisting of all *Whittaker modules* for the pair (\mathfrak{m}, χ) .

Definition 4.1.2. A module $M \in U(\mathfrak{g})\text{-mod}$ is called a *Whittaker module* for (\mathfrak{m}, χ) if the Lie algebra \mathfrak{m} acts by the generalised character χ , or equivalently if the shifted Lie algebra \mathfrak{m}_χ acts locally nilpotently on M . The category of Whittaker modules is denoted $\text{Whit}_{\mathfrak{m}, \chi}(\mathfrak{g})$.

$$\text{Whit}_{\mathfrak{m}, \chi}(\mathfrak{g}) := \{M \in U(\mathfrak{g})\text{-mod} : \forall m \in M, y \in \mathfrak{m}, \exists n > 0 \text{ such that } (y - \chi(y))^n m = 0\}$$

A *Whittaker vector* in a Whittaker module M is a vector $m \in M$ on which \mathfrak{m} acts strictly by the character χ . The space of Whittaker vectors of M is denoted $\text{Wh}(M)$.

$$\text{Wh}(M) := \{m \in M : \forall y \in \mathfrak{m}, (y - \chi(y))m = 0\}$$

Lemma 4.1.3. For any $M \in U(\mathfrak{g}, e)\text{-mod}$, $Q_\chi \otimes_{U(\mathfrak{g}, e)} M$ is a Whittaker module for (\mathfrak{m}, χ) . Furthermore, for a Whittaker module $M \in \text{Whit}_{\mathfrak{m}, \chi}(\mathfrak{g})$, the space of Whittaker vectors $\text{Wh}(M)$ is naturally a $U(\mathfrak{g}, e)$ -module.

Proof. To prove the first claim, it suffices to check that $y - \chi(y)$ acts locally nilpotently on Q_χ for all $y \in \mathfrak{m}$. Since \mathfrak{m} is strictly negatively graded with respect to the good grading Γ , it acts locally nilpotently on the Lie algebra \mathfrak{g} . The induced adjoint action on $U(\mathfrak{g})$ is also locally nilpotent, which can be seen by induction on the length in the PBW filtration. Hence for a sufficiently large $n > 0$, one can commute $(y - \chi(y))^n u = u(y - \chi(y))^n \in U(\mathfrak{g})\mathfrak{m}_\chi$, and so \mathfrak{m}_χ acts locally nilpotently on Q_χ .

To prove the second claim, it suffices to note that for $m \in \text{Wh}(M)$,

$$(y + U(\mathfrak{g})\mathfrak{m}_\chi) \cdot m = y \cdot m + U(\mathfrak{g})\mathfrak{m}_\chi \cdot m = y \cdot m$$

and so lifting $y + U(\mathfrak{g})\mathfrak{m}_\chi \in U(\mathfrak{g}, e)$ to $y \in U(\mathfrak{g})$ results in a well-defined action of $U(\mathfrak{g}, e)$ on M . \square

Theorem 4.1.4 (The Skryabin equivalence). [Pre1, Appendix 1, due to Skryabin]

The Skryabin functor is an equivalence of categories

$$Q_\chi \otimes_{U(\mathfrak{g}, e)} -: U(\mathfrak{g}, e)\text{-mod} \rightarrow \text{Whit}_{\mathfrak{m}, \chi}(\mathfrak{g})$$

with quasi-inverse functor

$$\text{Wh} : \text{Whit}_{\mathfrak{m}, \chi}(\mathfrak{g}) \rightarrow U(\mathfrak{g}, e)\text{-mod}.$$

Proof. [GG, Theorem 6.1] We begin by showing that $\text{Wh}(Q_\chi \otimes_{U(\mathfrak{g}, e)} M) \simeq M$ for all modules $M \in U(\mathfrak{g}, e)\text{-mod}$, and to this end note that $\text{Wh}(Q_\chi \otimes_{U(\mathfrak{g}, e)} M) \simeq H^0(\mathfrak{m}, (Q_\chi \otimes_{U(\mathfrak{g}, e)} M))$, where $Q_\chi \otimes_{U(\mathfrak{g}, e)} M$ is regarded as a \mathfrak{m} -module with action twisted by χ . The proof then proceeds in a similar manner as the proof of theorem 2.2.17: by demonstrating an isomorphism on the associated graded modules.

Claim. Assume that V is generated by a finite-dimensional subspace V_0 , and that V is therefore a filtered $U(\mathfrak{g}, e)$ -module with filtration $F_n V := (F_n U(\mathfrak{g}, e)) \cdot V_0$. Then

$$\mathrm{gr} H^0(\mathfrak{m}, Q_\chi \otimes_{U(\mathfrak{g}, e)} V) \simeq H^0(\mathfrak{m}, \mathrm{gr}(Q_\chi \otimes_{U(\mathfrak{g}, e)} V)) \simeq \mathrm{gr} V,$$

and all higher cohomology vanishes, i.e. $H^i(\mathfrak{m}, Q_\chi \otimes_{U(\mathfrak{g}, e)} V) = 0$ for all $i > 0$.

The second isomorphism follows from theorem 2.2.13 and lemma 2.2.18:

$$\mathrm{gr}(Q_\chi \otimes_{U(\mathfrak{g}, e)} V) \simeq \mathrm{gr} Q_\chi \otimes_{\mathrm{gr} U(\mathfrak{g}, e)} \mathrm{gr} V \simeq (\mathbb{C}[M'] \otimes \mathrm{gr} U(\mathfrak{g}, e)) \otimes_{\mathrm{gr} U(\mathfrak{g}, e)} \mathrm{gr} V \simeq \mathbb{C}[M'] \otimes \mathrm{gr} V.$$

The first isomorphism follows directly by noting that $Q_\chi \otimes_{U(\mathfrak{g}, e)} V$ is positively-graded and \mathfrak{m} is strictly-negatively graded, and then following through the proof of theorem 2.2.19 *mutatis mutandis*.

To prove the theorem, it show to prove that for any $V \in \mathrm{Whit}_{\mathfrak{m}, \chi}$, the canonical map

$$\gamma: Q_\chi \otimes_{U(\mathfrak{g}, e)} \mathrm{Wh}(V) \rightarrow V, \quad (u + U(\mathfrak{g})\mathfrak{m}_\chi) \otimes m \mapsto u \cdot m$$

is an isomorphism. Consider an exact sequence of $U(\mathfrak{g})$ -modules

$$0 \longrightarrow V' \longrightarrow Q_\chi \otimes_{U(\mathfrak{g}, e)} \mathrm{Wh}(V) \xrightarrow{\gamma} V \longrightarrow V'' \longrightarrow 0. \quad (4.1)$$

To see γ is injective, note that $\mathrm{Wh}(V') = V' \cap \mathrm{Wh}(Q_\chi \otimes_{U(\mathfrak{g}, e)} \mathrm{Wh}(V)) = V' \cap \mathrm{Wh}(V) = 0$, where the first equality follows from the fact that V' is a submodule, the second equality follows from the first part of this theorem, and the third follows from the definition of γ . But $\mathrm{Wh}(V)$ cannot be zero without V also being zero, hence γ is injective.

To prove γ is surjective, we aim to show that $V'' = 0$. The short exact sequence (in view of the fact that $V' = 0$) eq. (4.1) gives rise to a long exact sequence on cohomology

$$0 \longrightarrow H^0(\mathfrak{m}, Q_\chi \otimes_{U(\mathfrak{g}, e)} \mathrm{Wh}(V)) \xrightarrow{\gamma^*} H^0(\mathfrak{m}, V) \longrightarrow H^0(\mathfrak{m}, V'') \longrightarrow H^1(\mathfrak{m}, Q_\chi \otimes_{U(\mathfrak{g}, e)} V) = 0,$$

where the last equality follows from the above claim. Re-writing this sequence as an exact sequence of Whittaker vectors yields

$$0 \longrightarrow \mathrm{Wh}(Q_\chi \otimes_{U(\mathfrak{g}, e)} \mathrm{Wh}(V)) \xrightarrow{\gamma^*} \mathrm{Wh}(V) \longrightarrow \mathrm{Wh}(V'') \longrightarrow 0.$$

But the first two terms are equal, and hence γ^* is an isomorphism and $V'' = 0$, completing the proof that γ is surjective and therefore an isomorphism. \square

4.2 The definition of category \mathcal{O} for W -algebras

We now seek to define an analogue of category \mathcal{O} for the W -algebras $U(\mathfrak{g}, e)$. To do this, we need to make a choice of parabolic subalgebra \mathfrak{p} satisfying certain conditions. Let \mathfrak{p} be a parabolic subalgebra of \mathfrak{g} satisfying the two conditions:

1. e is a distinguished nilpotent in the Levi subalgebra \mathfrak{l} of \mathfrak{p} , i.e. the centraliser $\mathfrak{z}_{\mathfrak{l}}(e)$ contains no semisimple elements not lying in $\mathfrak{z}(\mathfrak{l})$;
2. \mathfrak{p} contains a fixed maximal torus \mathfrak{t} of the centraliser $\mathfrak{z}_{\mathfrak{g}}(e)$.

The choice of parabolic \mathfrak{p} allows one to define a pre-order on the weights of \mathfrak{t} as follows: $\lambda \geq \mu$ if and only if $\lambda - \mu$ is a linear combination of the weights of \mathfrak{t} acting on \mathfrak{p} .

Example 4.2.1. When $e = 0$, then \mathfrak{p} can be taken to be a Borel subalgebra \mathfrak{b} with Levi a fixed Cartan subalgebra \mathfrak{h} . As the centraliser $\mathfrak{z}_{\mathfrak{g}}(e) = \mathfrak{g}$, the maximal torus \mathfrak{t} can be taken to be \mathfrak{h} . The pre-order is the classical partial order on weights: $\lambda \geq \mu$ if and only if $\lambda - \mu$ is a positive linear combination of simple roots.

Example 4.2.2. When e is a regular nilpotent in \mathfrak{g} , then \mathfrak{p} can be taken to be \mathfrak{g} itself with Levi subalgebra \mathfrak{g} . The maximal torus is then just $\mathfrak{t} = \{0\}$. The only weight \mathfrak{t} is the zero weight, and the pre-order is trivial.

Example 4.2.3. Let $\mathfrak{g} = \mathfrak{sl}_3$ and $e = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. The parabolic subalgebra \mathfrak{p} , Levi subalgebra \mathfrak{l} and maximal torus \mathfrak{t} can be taken to be

$$\mathfrak{p} = \begin{pmatrix} * & * & * \\ * & * & * \\ 0 & 0 & * \end{pmatrix} \quad \mathfrak{l} = \begin{pmatrix} * & * & 0 \\ * & * & 0 \\ 0 & 0 & * \end{pmatrix} \quad \mathfrak{t} = \left\{ \begin{pmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & -2a \end{pmatrix} : a \in \mathbb{C} \right\}$$

As \mathfrak{t} is a one-dimensional, its weights are just complex numbers. Further, the set of weights of \mathfrak{t} acting on \mathfrak{p} is the set $\{0, 3\}$.

There is furthermore an embedding $U(\mathfrak{t}) \hookrightarrow U(\mathfrak{g}, e)$, and hence any $U(\mathfrak{g}, e)$ -module can be decomposed into its generalised weight spaces with respect to \mathfrak{t} . This can be seen either by choosing a \mathfrak{t} -invariant Premet subalgebra \mathfrak{m} , and hence the image of $U(\mathfrak{t})$ in $U(\mathfrak{g})/U(\mathfrak{g})\mathfrak{m}_{\chi}$ can be seen to be $\text{ad } \mathfrak{m}$ -invariant, or one can prove it for any choice of Premet subalgebra by a more careful argument, as in [BGK, Theorem 3.3]. This allows one to formulate a version of category \mathcal{O} for W -algebras.

Definition 4.2.4. Given a nilpotent element $e \in \mathfrak{g}$ with a compatible choice of parabolic subalgebra \mathfrak{p} and maximal torus \mathfrak{t} in $\mathfrak{z}(e)$ as above, we define a number of versions of *category \mathcal{O}* for the W -algebra $U(\mathfrak{g}, e)$. Each is a full subcategory of the finitely-generated modules in $U(\mathfrak{g}, e)$ -mod satisfying certain conditions. The weakest version is denoted $\tilde{\mathcal{O}}(e, \mathfrak{p})$, and consists of those modules M satisfying

1. the weights of M are contained in a finite union of sets of the form $\{\mu : \mu \leq \lambda\}$.

Another subcategory, denoted $\mathcal{O}(e, \mathfrak{p})$, is defined as consisting of modules which further satisfy the condition

2. \mathfrak{t} acts on M with finite-dimensional generalised weight spaces.

For each of these versions of category \mathcal{O} , we define a further subcategory consisting of those modules on which \mathfrak{t} acts semi-simply, and denote them $\tilde{\mathcal{O}}^{\mathfrak{t}}(e, \mathfrak{p})$ and $\mathcal{O}^{\mathfrak{t}}(e, \mathfrak{p})$, respectively. Note that $\mathcal{O}^{\mathfrak{t}}(e, \mathfrak{p}) = \tilde{\mathcal{O}}^{\mathfrak{t}}(e, \mathfrak{p}) \cap \mathcal{O}(e, \mathfrak{p})$.

Remark 4.2.5. This forms an analogue of the classical BGG category \mathcal{O} defined at the beginning of this chapter. Condition 2 here corresponds to condition 2 in that definition, and condition 1 here corresponds to both conditions 1 and 3.

Example 4.2.6. If $e = 0$ and \mathfrak{b} is a chosen Borel subalgebra, then $\mathcal{O}^{\mathfrak{t}}(0, \mathfrak{b})$ is just the classical category \mathcal{O} for the chosen Borel.

Example 4.2.7. If $e_{\text{reg}} \in \mathfrak{g}$ is a regular nilpotent, then $\mathfrak{p} = \mathfrak{g}$ and $\mathcal{O}^{\mathfrak{t}}(e_{\text{reg}}, \mathfrak{g})$ is just the category of all finite-dimensional modules over $U(\mathfrak{g}, e_{\text{reg}}) \simeq Z(\mathfrak{g})$:

$$\mathcal{O}^{\mathfrak{t}}(e_{\text{reg}}, \mathfrak{g}) \simeq Z(\mathfrak{g})\text{-fmod}.$$

4.3 Generalised Whittaker modules and a category equivalence

The Skryabin equivalence gives an equivalence between the category of all modules for a W -algebra and the category of Whittaker modules for the Premet subalgebra \mathfrak{m} and character χ . In [Los1, Theorem 4.1], Losev introduces the concept of a *generalised Whittaker module*, and proves an equivalence between the categories \mathcal{O} for W -algebras and appropriate categories of generalised Whittaker modules.

Let $e \in \mathfrak{g}$ be a nilpotent with good grading $\mathfrak{g} = \bigoplus_{i \in \mathbb{Z}} \mathfrak{g}_i$. Given a choice of parabolic \mathfrak{p} with e regular in the Levi \mathfrak{l} , one can construct an analogue of the Premet subalgebra $\mathfrak{m} \subseteq \mathfrak{g}$. Let $\mathfrak{l} = \bigoplus_{i \in \mathbb{Z}} \mathfrak{l}_i$ be the induced grading on \mathfrak{l} coming from \mathfrak{g} , and choose a Premet subalgebra $\underline{\mathfrak{m}} \subseteq \mathfrak{l}$ in the usual way. Note that this allows one to define the W -algebra $U(\mathfrak{l}, e) := (U(\mathfrak{l})/U(\mathfrak{l})\underline{\mathfrak{m}}_\chi)^{\text{ad } \underline{\mathfrak{m}}}$. One can then consider the nilradical \mathfrak{u} of \mathfrak{p} , where $\mathfrak{p} = \mathfrak{l} \oplus \mathfrak{u}$, and define the new subalgebra $\tilde{\mathfrak{m}} := \underline{\mathfrak{m}} \oplus \mathfrak{u}$, along with its shift $\tilde{\mathfrak{m}}_\chi$.

Definition 4.3.1. A $U(\mathfrak{g})$ -module M is a *generalised Whittaker module* for the pair (e, \mathfrak{p}) if $\tilde{\mathfrak{m}}$ acts on M by generalised character χ , or equivalently if $\tilde{\mathfrak{m}}_\chi$ acts on M by locally nilpotent endomorphisms. The full subcategory of all generalised Whittaker modules in $U(\mathfrak{g})\text{-mod}$ is denoted $\widetilde{\text{Whit}}_{e, \mathfrak{p}}$, while the full subcategory of $\widetilde{\text{Whit}}_{e, \mathfrak{p}}$ consisting of those modules on which \mathfrak{t} acts semi-simply is denoted $\widetilde{\text{Whit}}_{e, \mathfrak{p}}^{\mathfrak{t}}$.

The functor $(-)^{\tilde{\mathfrak{m}}_\chi}$, taking $\tilde{\mathfrak{m}}_\chi$ -invariants of the module M , gives a natural functor from $\widetilde{\text{Whit}}_{e, \mathfrak{p}}$ to $U(\mathfrak{l}, e)\text{-mod}$. This can be seen as $U(\mathfrak{l})$ acts naturally on $M^{\mathfrak{u}}$ for any $M \in \widetilde{\text{Whit}}_{e, \mathfrak{p}}$, and $M^{\tilde{\mathfrak{m}}_\chi} = (M^{\mathfrak{u}})^{\underline{\mathfrak{m}}_\chi}$, so there is a well-defined action of $U(\mathfrak{l})/U(\mathfrak{l})\underline{\mathfrak{m}}_\chi$. A module $M \in \widetilde{\text{Whit}}_{e, \mathfrak{p}}$ is said to be of *finite type* if $M^{\tilde{\mathfrak{m}}_\chi}$ is finite-dimensional when viewed as a $U(\mathfrak{l}, e)$ -module.

Proposition 4.3.2. [Los1, Proposition 4.2] *If $e \in \mathfrak{g}$ is a regular nilpotent in \mathfrak{l} , then a generalised Whittaker module $M \in \widetilde{\text{Whit}}_{e, \mathfrak{p}}$ is of finite type if and only if the action of $Z(\mathfrak{g}) \subseteq U(\mathfrak{g})$ on M is locally finite. In particular, this holds for any nilpotent in type A.*

Definition 4.3.3. The category of all finite-type modules in $\widetilde{\text{Whit}}_{e, \mathfrak{p}}$ is denoted $\text{Whit}_{e, \mathfrak{p}}$, and the subcategory of finite-type modules on which \mathfrak{t} acts semi-simply is denoted $\text{Whit}_{e, \mathfrak{p}}^{\mathfrak{t}}$.

Theorem 4.3.4. [Los1, Theorem 4.1] *There is an equivalence of categories*

$$K: \widetilde{\text{Whit}}_{e, \mathfrak{p}} \rightarrow \tilde{\mathcal{O}}(e, \mathfrak{p}).$$

This furthermore induces equivalences of subcategories

$$K: \widetilde{\text{Whit}}_{e, \mathfrak{p}}^{\mathfrak{t}} \rightarrow \tilde{\mathcal{O}}^{\mathfrak{t}}(e, \mathfrak{p})$$

$$K: \text{Whit}_{e, \mathfrak{p}} \rightarrow \mathcal{O}(e, \mathfrak{p})$$

$$K: \text{Whit}_{e, \mathfrak{p}}^{\mathfrak{t}} \rightarrow \mathcal{O}^{\mathfrak{t}}(e, \mathfrak{p}).$$

The proof of theorem 4.3.4 uses Losev's techniques of completions of deformation quantisations of Poisson algebras, developed in, e.g. [Los2]. The second main result of this thesis is a generalisation of the above theorem, so we shall develop Losev's machinery here, with an eye to applying it in the proof of this theorem and its generalisation.

4.3.1 Deformation quantisations of Poisson algebras

Losev's results are geometric in nature, and rely on the interpretation of W -algebras as *deformation quantisations* of certain commutative Poisson algebras.

Definition 4.3.5. Given a Poisson algebra A with Poisson bracket $\{\cdot, \cdot\}$, a *deformation quantisation* of A is an associative unital product $\star: A \otimes A \rightarrow A[[\hbar]]$ (often called a *star product*). Writing $f \star g = \sum_{k \geq 0} D_k(f, g) \hbar^{2k}$, and extending $\mathbb{C}[[\hbar]]$ -bilinearly to a product on $A[[\hbar]]$, a deformation quantisation \star is one which satisfies the following conditions for $f, g \in A$ (viewed as elements of $A[[\hbar]]$ by the natural inclusion):

1. \star is an associative binary product on $A[[\hbar]]$, continuous in the \hbar -adic topology;
2. $f \star g = fg + O(\hbar^2)$ (i.e. $D_0(f, g) = fg$), which implies that \star degenerates to the usual multiplication in A when $\hbar = 0$;
3. $[f, g] := f \star g - g \star f = \hbar^2 \{f, g\} + O(\hbar^4)$ (i.e. $D_1(f, g) - D_1(g, f) = \{f, g\}$), which means the \hbar^2 -term of \star encodes the Poisson bracket of A .

We shall also require the stronger condition that \star is a *differential* deformation quantisation, namely that:

4. For each k , $D_k(\cdot, \cdot)$ is a bidifferential operator of order at most k in each variable.

Remark 4.3.6. Usually, the star product is expanded as $f \star g := \sum_{k \geq 0} D_k(f, g) \hbar^k$, and the powers of \hbar in conditions 2 and 3 are halved (i.e. \hbar in place of \hbar^2 , and \hbar^2 in place of \hbar^4). The conventions here are used for better compatibility with the Kazhdan filtration.

Remark 4.3.7. The star product on the algebra $A[[\hbar]]$ can be used to introduce a new product on the Poisson algebra A , induced by the projection $A[[\hbar]] \rightarrow A$, $\hbar \mapsto 1$. Concretely, define the new associative product $\circ: A \otimes A \rightarrow A$ by $f \circ g := \sum_{k \geq 0} D_k(f, g)$. Denote the algebra A with this new algebra structure by \mathcal{A} .

Proposition 4.3.8. [Los2, Corollary 3.3.3] *The Rees algebra of the W -algebra $U(\mathfrak{g}, e)$, viewed as a filtered algebra with the Kazhdan filtration, is the unique deformation quantisation of the ring of functions on the Slodowy slice \mathcal{S}_χ , up to isomorphism.*

4.3.2 Completions of quantum algebras and Losev's machinery

To prove theorem 4.3.4, we need to introduce a technical theorem of Losev [Los1, Proposition 5.1], and to that end introduce some needed notation. In addition, the machinery of this proposition shall be needed to prove the second main result of this thesis.

Assume that all of the following hold:

- $\mathfrak{v} = \bigoplus_{i \in \mathbb{Z}} \mathfrak{v}_i$ is a graded finite-dimensional vector space on which a torus T acting by preserving the grading.
- $A := \text{Sym}(\mathfrak{v})$, with an induced grading $A = \bigoplus_{i \in \mathbb{Z}} A_i$ and induced action by T .
- \mathcal{A} is an algebra with the same underlying vector space as A , where the algebra structure comes from a T -invariant homogeneous star product.
- ω_1 is a symplectic form on \mathfrak{v}_1 (where $\omega_1(u, v)$ is the constant term of the commutator in \mathcal{A}), and η is a lagrangian subspace of \mathfrak{v}_1 .

- $\mathfrak{m} := \mathfrak{v} \oplus \bigoplus_{i \leq 0} \mathfrak{v}_i$.
- v_1, v_2, \dots, v_n is a homogeneous basis of \mathfrak{v} such that v_1, v_2, \dots, v_m form a basis of \mathfrak{m} . Further, let d_i be the degree of v_i and assume that d_1, d_2, \dots, d_m are increasing and that all v_i are T -semi-invariant.
- \mathcal{A}^\heartsuit is the subalgebra of $\mathbb{C}[[\mathfrak{v}^*]]$ consisting of elements of the form $\sum_{i \leq n} f_i$ for some n , where f_i is a homogeneous power series of degree i .
- \mathcal{A}^\heartsuit is the algebra \mathcal{A}^\heartsuit with multiplication coming from \mathcal{A} . Any element of \mathcal{A}^\heartsuit can be written as an infinite linear combination of monomials $v_{i_1} v_{i_2} \cdots v_{i_\ell}$ where $i_1 \geq \cdots \geq i_\ell$, and where $\sum_{j=1}^\ell d_{i_j} \leq c$ for some c . This gives us a filtration $F_c \mathcal{A}^\heartsuit$.
- θ is a co-character of T , and $\mathfrak{v}_{\geq 0}$ and $\mathfrak{v}_{>0}$ are the sums of positive and strictly-positive (respectively) $\text{ad } \theta$ -eigenspaces of \mathfrak{v} . We require that $\mathfrak{v}_{>0} \subseteq \mathfrak{m} \subseteq \mathfrak{v}_{\geq 0}$. Note that beyond this condition, these spaces are unrelated to the \mathbb{Z} -grading on the vector space \mathfrak{v} .
- $\mathcal{A}_{\geq 0}, \mathcal{A}_{>0}, \mathcal{A}_{\geq 0}^\heartsuit, \mathcal{A}_{>0}^\heartsuit$ are all defined analogously.
- $\mathcal{A}^\wedge := \lim A/\mathcal{A}\mathfrak{m}^k$, which has an injective algebra homomorphism $\mathcal{A}^\heartsuit \rightarrow \mathcal{A}^\wedge$.

Proposition 4.3.9. [Los1, Proposition 5.1] *Let \mathfrak{v} and A be as above, and \mathcal{A} and \mathcal{A}' be two different algebras coming from A as above with two different star products. Suppose there is a subspace $\mathfrak{v} \subseteq \mathfrak{v}(1)$ which is Lagrangian for both symplectic forms, and every element of A can be written as a finite sum of monomials in both \mathcal{A} and \mathcal{A}' . Then any homogeneous T -equivariant isomorphism $\Phi: \mathcal{A}^\heartsuit \rightarrow \mathcal{A}'^\heartsuit$ satisfying $\Phi(v_i) - v_i \in F_{d_i-2}A + (F_{d_i}A \cap \mathfrak{v}^2A)$ extends uniquely to a topological algebra isomorphism $\Phi: \mathcal{A}^\wedge \rightarrow \mathcal{A}'^\wedge$ with $\Phi(\mathcal{A}^\wedge \mathfrak{m}) = \mathcal{A}'^\wedge \mathfrak{m}$.*

Corollary 4.3.10. [Los1, Corollary 5.2] *The equivalence $\Phi: \mathcal{A}^\wedge \rightarrow \mathcal{A}'^\wedge$ of the above proposition induces equivalences $\Phi_*: \text{Whit}_{\mathfrak{m}}(\mathcal{A}) \rightarrow \text{Whit}_{\mathfrak{m}}(\mathcal{A}')$ and $\Phi_*: \text{Whit}_{\mathfrak{m}}^t(\mathcal{A}) \rightarrow \text{Whit}_{\mathfrak{m}}^t(\mathcal{A}')$, where again $\text{Whit}_{\mathfrak{m}}(\mathcal{A})$ is the category of all \mathcal{A} -modules on which \mathfrak{m} acts by locally nilpotent endomorphisms, and $\text{Whit}_{\mathfrak{m}}^t(\mathcal{A})$ is the subcategory on which \mathfrak{t} acts semi-simply. Furthermore, this functor commutes with taking \mathfrak{m} -invariants, i.e. $\Phi_*(M^{\mathfrak{m}}) = \Phi_*(M)^{\mathfrak{m}}$ for all $M \in \text{Whit}_{\mathfrak{m}}(\mathcal{A})$.*

Losev uses these technical results to construct an isomorphism between the completed universal enveloping algebra $U(\mathfrak{g})_\chi^\wedge$ and $U(\mathfrak{g}, e)_\chi^\wedge \hat{\otimes} \mathbf{A}_{V_0}^\wedge$, the completed tensor product of the W -algebra and the completed Weyl algebra $\mathbf{A}_{V_0}^\wedge$, where V is the symplectic leaf through e . He does this by making the following choices for the above objects. Recall that we start with a semisimple Lie algebra \mathfrak{g} with distinguished nilpotent element $e \in \mathfrak{g}$ forming part of an \mathfrak{sl}_2 -triple $\{e, h, f\}$.

1. $\mathfrak{v} := \mathfrak{g}_\chi = \{\xi - \chi(\xi) : \xi \in \mathfrak{g}\}$.
2. T is a connected maximal torus in the centraliser $Z_{\mathfrak{g}}(\{e, h, f\})$ with $h \in \mathfrak{t}$; θ is an arbitrary (generic) co-character $\theta \in \text{Hom}(\mathbb{C}^\times, T)$. This determines a parabolic \mathfrak{p} as the span of the positive eigenspaces of $\text{ad } \theta$.
3. Choosing $m > 2 + 2d$, where d is the maximum eigenvalue of $\text{ad } h$ in \mathfrak{g} , define the grading on \mathfrak{v} by $\mathfrak{v}_i = \{\xi \in \mathfrak{g} : (h - m\theta)\xi = (i - 2)\xi\}$. This is just the Kazhdan grading with a shift by m so that everything in the nilradical of \mathfrak{p} has negative grading. The algebra \mathfrak{m} corresponds to a choice of $\tilde{\mathfrak{m}}_\chi$.

4. $\mathcal{A} := U(\mathfrak{g})$ and $\mathcal{A}' := U(\mathfrak{g}, e) \otimes \mathbf{A}_V$, where $V = [\mathfrak{g}, f]$ is the symplectic leaf in \mathfrak{g} through e , and \mathbf{A}_V is its Weyl algebra.

The map $\Phi: U(\mathfrak{g})^\vee \rightarrow (U(\mathfrak{g}, e) \otimes \mathbf{A}_V)^\vee$ exists by [Los1, Lemma 5.3], and comes from an application of the Luna slice theorem, and the identification of these spaces as the quotients of the \mathbb{C}^\times -finite parts of the quantum algebras

$$\mathbb{C}[\mathfrak{g}^*]_\chi^\wedge[[\hbar]] \quad \text{and} \quad \mathbb{C}[\mathcal{S}_\chi]_\chi^\wedge[[\hbar]] \widehat{\otimes}_{\mathbb{C}[[\hbar]]} \mathbb{C}[V^*]_0^\wedge[[\hbar]]$$

by the ideal generated by $\hbar - 1$.

Proof of theorem 4.3.4. To prove the theorem, we examine the category equivalences of corollary 4.3.10. Note that $\text{Whit}_m(\mathcal{A}) = \widetilde{\text{Whit}}_{e, \mathfrak{p}}$ for \mathfrak{p} the parabolic subalgebra determined by θ , and similarly the subcategories of t -semisimple modules are equal, $\text{Whit}_m^t(\mathcal{A}) = \widetilde{\text{Whit}}_{e, \mathfrak{p}}^t$.

It remains to construct an equivalence between $\text{Whit}_m(\mathcal{A}')$ and $\widetilde{\mathcal{O}}(e, \mathfrak{p})$; let this be given by the invariant functor with respect to $m \cap V$, which can be seen to be a Lagrangian subspace of V by applying property GG5 of good gradings in the Levi subalgebra.

$$\begin{aligned} K' : \text{Whit}_m(\mathcal{A}') &\rightarrow \widetilde{\mathcal{O}}(e, \mathfrak{p}) \\ M &\mapsto M^{m \cap V} \end{aligned}$$

The desired functor $K: \widetilde{\text{Whit}}_{e, \mathfrak{p}} \rightarrow \widetilde{\mathcal{O}}(e, \mathfrak{p})$ is the composition of functors $K' \circ \Phi_*$. It preserves the necessary subcategories by corollary 4.3.10. \square

4.4 Equivalences between categories \mathcal{O}

We shall use a similar argument to that of Losev to prove an equivalence between the category $\widetilde{\mathcal{O}}(e_2, \mathfrak{p}')$ and an appropriate subcategory of Whittaker vectors in $\widetilde{\mathcal{O}}(e_1, \mathfrak{p})$ for a pair of nilpotent elements $e_1 \leq e_2$ in a Lie algebra of type A where the partial ordering considered is the *refinement ordering*. We expand upon the reductions produced in chapter 3 to produce these.

Definition 4.4.1. Let $\lambda = (\lambda_1, \dots, \lambda_k)$ and $\mu = (\mu_1, \dots, \mu_\ell)$ be two partitions of n satisfying $\lambda_1 \geq \dots \geq \lambda_k > 0$ and $\mu_1 \geq \dots \geq \mu_\ell > 0$. The *refinement ordering* is the partial ordering on the set of partitions where $\lambda \geq \mu$ if and only if μ is a *refinement* of λ , i.e. there exists a partition $\nu_1, \nu_2, \dots, \nu_k$ of the set $\{1, \dots, \ell\}$ such that $\lambda_i = \sum_{j \in \nu_i} \mu_j$ for every i .

We also call the partial ordering on nilpotent orbits in \mathfrak{sl}_n induced by this ordering the *refinement ordering*. Concretely, let \mathcal{O}_1 and \mathcal{O}_2 be two nilpotent orbits in \mathfrak{sl}_n corresponding to partitions λ and μ , respectively. We say that $\mathcal{O}_1 \geq \mathcal{O}_2$ in the refinement ordering if $\lambda \geq \mu$ in the refinement ordering.

Remark 4.4.2. Note that the dominance ordering is a *refinement* of the refinement ordering; this means that $\lambda \geq \mu$ in the dominance ordering whenever this holds in the refinement ordering. In general, it can be the case that $\lambda \geq \mu$ in the dominance ordering, but $\lambda \not\geq \mu$ in the refinement ordering. For example, $(3, 1) > (2, 2)$ in the dominance ordering, but not in the refinement ordering.

Choose a pair of \mathfrak{sl}_2 -triples $\{e_1, h_1, f_1\}$ and $\{e_2, h_2, f_2\}$ for each of the nilpotents elements e_1 and e_2 . As in section 4.3.2, we seek to make a set of choices which satisfy the hypotheses of proposition 4.3.9.

1. Define $\mathfrak{v} := \{\xi - \chi_2(\xi) : \xi \in \mathfrak{z}(e_1)\}$.
2. Note that since $e_1 < e_2$ in the refinement ordering, the pair of Levi subalgebras \mathfrak{l}_1 and \mathfrak{l}_2 , chosen such that e_1 and e_2 are regular their respective Levi subalgebras, can further be chosen so that \mathfrak{l}_1 is a subalgebra of \mathfrak{l}_2 . As a result, a maximal connected torus T in the centraliser $Z_{\mathfrak{g}}(\{e_2, h_2, f_2\})$ can be chosen which preserves \mathfrak{v} . Choose an arbitrary generic co-character $\theta_2 \in \text{Hom}(\mathbb{C}^\times, T)$.
3. Choosing $m > 2 + 2d$, where d is the maximum eigenvalue of $\text{ad } h_2$ on \mathfrak{g} , we define the grading on \mathfrak{v} to be give by $\mathfrak{v}_i = \{\xi \in \mathfrak{v} : (\text{ad } h_2 - m\theta)\xi = (i - 2)\xi\}$.
4. Define $\mathcal{A} := U(\mathfrak{g}, e_1)$ and $\mathcal{A}' = U(\mathfrak{g}, e_2) \otimes \mathbf{A}_V$, where $V = [\mathfrak{g}, f_2] \cap \mathfrak{z}(e_1)$ is the symplectic leaf in \mathcal{S}_{e_1} passing through e_2 .

Lemma 4.4.3. *These choices satisfy the hypotheses of proposition 4.3.9.*

Proof. It is clear from the construction that \mathfrak{v} is a finite-dimensional graded vector space, and that the torus T preserves the grading. Since $\mathcal{A} = U(\mathfrak{g}, e_1)$ is a deformation quantisation of the Slodowy slice \mathcal{S}_{χ_1} , it follows that it comes from a T -invariant homogeneous star product on $\mathbb{C}[\mathcal{S}_{\chi_1}] = \text{Sym}(\mathfrak{z}(e))$. ω_1 is again a symplectic form on \mathfrak{v}_1 , and \mathfrak{m} is defined similarly. It remains no demonstrate that \mathcal{A} and \mathcal{A}' are both algebras coming from $A = \text{Sym}(\mathfrak{z}(e))$ with different star products.

Consider the subalgebras \mathfrak{m}_1 and \mathfrak{m}_2 constructed in chapter 3. As the moment map pre-image $\mu_i^{-1}(\chi_i) = \chi_i + \mathfrak{m}_i^{*,\perp}$, this gives the commutative diagram.

$$\begin{array}{ccc}
 \mathfrak{g}^* & \xlongequal{\quad} & \mathfrak{g}^* \\
 \uparrow & & \uparrow \\
 \chi_2 + \mathfrak{m}_2^{*,\perp} & \xhookrightarrow{\quad \iota \quad} & \chi_1 + \mathfrak{m}_1^{*,\perp} \\
 \uparrow & & \uparrow \\
 \mathcal{S}_{\chi_2} & \xhookrightarrow{\quad \varphi \quad} & \mathcal{S}_{\chi_1}
 \end{array} \tag{4.2}$$

Here, the vertical maps $\chi_i + \mathfrak{m}_i^{*,\perp} \hookrightarrow \mathfrak{g}^*$ and $\chi_i + \mathfrak{m}_i^{*,\perp} \twoheadrightarrow \mathcal{S}_{\chi_i}$ are the natural maps coming from the expression of \mathcal{S}_{χ_i} as a Hamiltonian reduction of \mathfrak{g}^* , and the maps $\mathcal{S}_{\chi_i} \hookrightarrow \chi_i + \mathfrak{m}_i^{*,\perp}$ are the maps which come from the natural expression $\mathcal{S}_{\chi_i} = \kappa(e_i + \mathfrak{z}(f_i)) \subseteq \mathfrak{g}^*$. The map ι comes from the embedding of \mathfrak{m}_1 as a subspace of \mathfrak{m}_2 by the decomposition $\mathfrak{m}_2 = \mathfrak{m}_1 \rtimes \mathfrak{k}$, and the map φ is defined as the composition of the necessary morphisms. Note that the embedding $\varphi: \mathcal{S}_{\chi_2} \hookrightarrow \mathcal{S}_{\chi_1}$ is a transverse slice to the symplectic leaf $V = [\mathfrak{g}, e_2] \cap \mathfrak{z}(e_1) \subseteq \mathcal{S}_{\chi_1}$. \square

Lemma 4.4.4. *With these choices, there is a map $\Phi: U(\mathfrak{g}, e_1)^\heartsuit \rightarrow (U(\mathfrak{g}, e_2) \otimes \mathbf{A}_V)^\heartsuit$.*

Proof. [Los2, Theorem 3.3.1] Consider Losev's equivariant Slodowy slices: $X_i := G \times \mathcal{S}_{\chi_i}$, where $G = SL_n$. The embedding $\mathcal{S}_{\chi_2} \hookrightarrow \mathcal{S}_{\chi_1}$ extends to an embedding $X_2 \hookrightarrow X_1$. Note that both X_i have an action of $\tilde{G} := G \times \mathbb{C}^\times \times G_0$, where G_0 is the subgroup of $Z_G(e_2, h_2, f_2)$ which preserves the grading, defined as follows:

$$g \cdot (g_1, \alpha) = (gg_1, \alpha) \quad t \cdot (g_1, \alpha) = (g_1 \gamma(t)^{-1}, t \cdot \alpha) \quad g_0 \cdot (g_1, \alpha) = (g_1 g_0^{-1}, g_0 \alpha)$$

Both X_i are stable under the action of \tilde{G} . Let $x = (1, \chi_2)$ and note that since $\tilde{G} \cdot x = G \cdot x$, which implies that $\tilde{G} \cdot x$ is closed. The stabiliser of x is $\{(g_0 \gamma(t), t, g_0) : t \in \mathbb{C}^\times, g_0 \in G_0\}$, which can be identified with $G_0 \times \mathbb{C}^\times$.

The symplectic subspace $T_x X_2 \subseteq T_x X_1$ has an orthogonal complement identified with V , resulting in a $(G_0 \times \mathbb{C}^\times)$ -equivariant symplectomorphism $\psi: T_x X_1 \rightarrow T_x X_2 \oplus V^*$. The Fedosov star products on $\mathbb{C}[X_1][[\hbar]]$ and $\mathbb{C}[X_2 \times V^*][[\hbar]]$ are both differential, so they extend to the completions $\mathbb{C}[X_1]_{Gx}^\wedge[[\hbar]]$ and $\mathbb{C}[X_2 \times V^*]_{Gx}^\wedge[[\hbar]]$.

Now, proceeding as in [Los2, Theorem 3.3.1], the map ψ and the Luna slice theorem can be used to produce a \tilde{G} -equivariant isomorphism $\Phi_\hbar: \mathbb{C}[X_1]_{Gx}^\wedge[[\hbar]] \rightarrow \mathbb{C}[X_2 \times V^*]_{Gx}^\wedge[[\hbar]]$. Taking G -invariants induces a map $\Phi_\hbar: \mathbb{C}[\mathcal{S}_{\chi_1}]_{Gx}^\wedge[[\hbar]] \rightarrow \mathbb{C}[\mathcal{S}_{\chi_2} \times V^*]_{Gx}^\wedge[[\hbar]]$, and restricting to the \mathbb{C}^\times -finite parts produces the desired map $\Phi: U(\mathfrak{g}, e_1)^\heartsuit \rightarrow (U(\mathfrak{g}, e_2) \otimes \mathbf{A}_V)^\heartsuit$. \square

Combining lemmas 4.4.3 and 4.4.4 with proposition 4.3.9 and corollary 4.3.10 completes the proof of the following theorem, forming an analogue of Losev's results for our case.

Theorem 4.4.5. *There is an equivalence of categories*

$$K: \text{Whit}_m^t(U(\mathfrak{g}, e_1)) \rightarrow \mathcal{O}^t(e_2, \mathfrak{p}_2),$$

where $\text{Whit}_m^t(U(\mathfrak{g}, e_1))$ is a full subcategory of $\mathcal{O}^t(e_1, \mathfrak{p}_1)$.

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